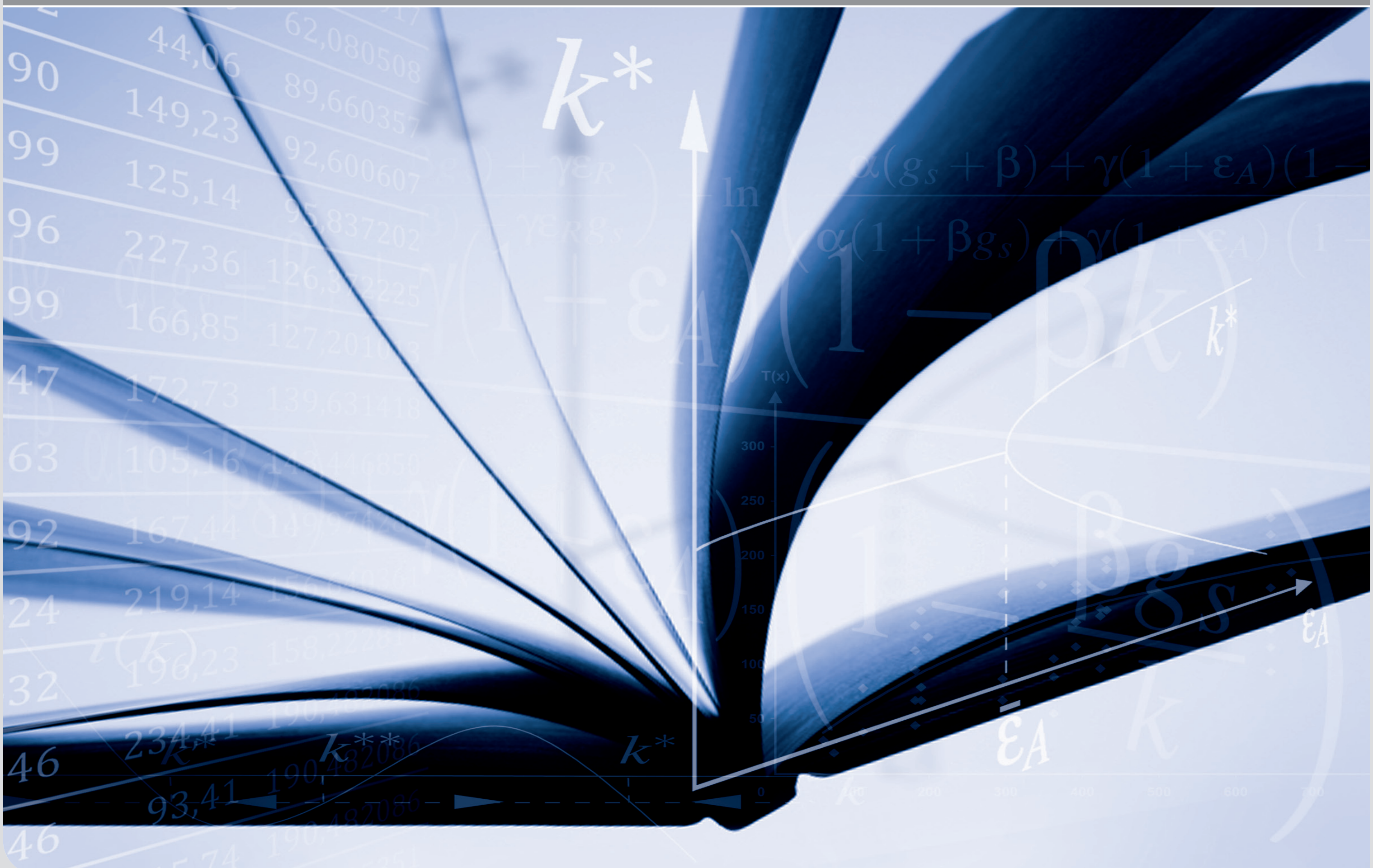
Karlsruher Institut für Technologie

# Analysis of the 2025 Bundestag elections. Part 1 of 4: Imperfection of the electoral reform

by Andranik S. Tangian

No. 167 | JUNE 2025

WORKING PAPER SERIES IN ECONOMICS



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**Imprint**


Karlsruher Institute of Technology (KIT)
Department of Economics and Management
Institute of Economics (ECON)

Kaiserstraße 12
76131 Karlsruhe

KIT – The Research University in the Helmholtz Association

Working Paper Series in Economics
**No. 167**, June 2025

ISSN 2190-9806

econpapers.wiwi.kit.edu

Institute of Economic Theory and Operations Research

Karlsruhe Institute of Technology

# Analysis of the 2025 Bundestag Elections. 1/4. Imperfection of the Electoral Reform

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Working paper Nr. 167

June 2025

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## Abstract

This is the first of four articles on the 2025 German federal elections, continuing our analysis of the 2009, 2013, 2017 and 2021 elections. We begin with the 2023/24 electoral reform, which aimed to curb the uncontrolled growth of the Bundestag caused by political developments not envisaged in the original election rules. The reform fixes the size of the Bundestag at 630 members and introduces limits to parties' mandates at the level of federal states *(Länder)*. All this makes the proportional allocation of Bundestag seats to parties less accurate and skews the balance between the two concepts implemented in the German mixed-member proportional representation system — the descriptive one (parliament consists of local representatives in order to 'mirror' the society) and the agent one (parliament consists of credible political experts from political parties) — in favor of the agent concept at the expense of the descriptive one.

We show that the accuracy of Bundestag seat allocation to parties can be improved by applying modern discrete optimization techniques instead of the currently used historical Sainte-Laguë/Webster method. The balance between the two concepts of representation can be restored by replacing the official two-tier distribution of Bundestag seats between federal state party associations with that computed directly in one step. Finally, all apportionment problems can be completely solved by introducing adjustment vote weights. All these devices are illustrated using hypothetical redistributions of Bundestag seats.



**Keywords:** Representative democracy, elections, theory of voting, proportional representation.

**JEL Classification:** D71


## Contents

# 1 Introduction

This is the first of four papers on the 2025 German federal elections continuing our analysis of the 2009, 2013, 2017 and 2021 elections [Tangian 2014, 2017, 2020, 2022a–d]. We begin with the 2023/24 reform [Bundesvervassungsgericht 2024, Federal Ministry of the Interior 2024, Siefken 2024], which aimed to curb the uncontrolled growth of the Bundestag, the second largest parliament in the world after China.

The German two-vote electoral system embodies two major concepts of political representation coined during the American and French Revolutions. Before the 2023/24 reform, the situation was as follows. The *descriptive concept* (leading to proportional representation) — that is, the parliament portrays the society in miniature[1] — was implemented in the first vote *(Erststimme)*, with which 299 constituencies (*Wahlkreise)*, of about 250,000 citizens each, elected local representatives by simple plurality. These so-called *direct mandate* holders took 299 Bundestag seats. The *agent concept* (leading to majoritarianism) — that is, the parliament is a committee of political experts who make majority decisions as the people's trustees and not simply as their fellow countrymen[2] — was embodied in the second vote *(Zweitstimme)* for a party. The second vote served two purposes: (1) to qualify parties receiving at least 5% of the second votes nationwide for the Bundestag seats (this restriction did not apply to parties with at least three direct mandates, to the four German parties of ethnical minorities with a special status, and to party-independent constituency election winners), and (2) to determine the proportion of party factions in the Bundestag. To meet the second goal, another 299 regular Bundestag seats were allocated to parties, which were *adjustment* or *leveling* seats. If the required proportion between party factions was unattainable to within the rounding accuracy of 0.5 seat, some extra adjustment seats had to be added [Bundestag 2023]. Thus, the ratio of regular Bundestag seats 299 : 299 reflected the intended 1 : 1 balance between the descriptive and agent concepts of representation.

The growing number of qualified parties and overhang mandates *(Überhangsmandate)* — the direct mandates allotted to the parties that received too few second votes, required more and more adjustment seats. To retain the proportionality between Bundestag party factions and their second votes within the accuracy of 0.5 seat, the 2002, 2005, 2009, 2013, 2017 and 2021 Bundestags were enlarged by 5, 16, 24, 33, 111 and 138 additional seats, respectively, having increased the size of the 2021 Bundestag up to 736 members — and this despite the fact that three CSU mandates were 'tolerably' not adjusted, otherwise the 2021 Bundestag would have as many as 786 members. The Bundestag was becoming increasingly expensive for taxpayers: its annual budget was already approaching a billion euros [Finthammer 2018]. In 2016, Norbert Lammert, then president of the Bundestag (2005–2017), proposed limiting it to 630 members, distributing mandates according to quotas for each of the German federal states (*Länder* — there are 16 of them), which were to be proportional to their electorate [Roßner 2016]. In October 2019, after predictions that the next Bundestag could exceed 800 seats, did some 100 German experts in constitutional law write an open letter suggesting to constrain its size by reducing the number of effective constituencies, and the Bundestag vice-president, Thomas Oppermann, called for such a reform without delay [Spiegel online 2019, Zeit online 2019].

The 2023/24 electoral reform, which is not yet finalized [Bürgerrat 2025], retains the 299 constituencies, as well as the first and second votes with their functions (the 5% qualification threshold was questioned but remains in place for now). What is new is that the Bundestag size is fixed at exactly 630 seats and that certain direct mandates can be reduced in order to avoid both overhangs and extra adjustment seats.

[1]The descriptive concept was defended in America by John Adams (1735–1826), one of the key Founding Fathers, the first Vice President and the Second President of the United States from 1797–1801. In France, the same viewpoint was shared by Honore Gabriel Riqueti, comte de Mirabeau (1749–1791), a statesman, a moderate revolutionary and promoter of a British-like constitutional monarchy [Manin 1997, p. 111].

[2]The agent concept was promoted by American Federalists, particularly by Alexander Hamilton (1755?–1804), one of key Founding Father of the United States, and James Madison (1751–1836), the fourth President of the USA from 1809–1817. In France, the concept of political representative as professional was developed by Abbé Emmanuel Joseph Sieyès (1748–1836), clergyman and political writer [Manin 1997, pp. 2–3, 129–131].

The allocation of Bundestag seats to parties is calculated in two steps: (1) as before, the Bundestag seats are distributed between parties in proportion to their nationwide second votes, and (2) the Bundestag seats of every party are distributed between federal states in proportion to the second votes the party receives there, setting federal state restrictions. Now, not all constituency election winners receive direct mandates automatically. If the number of constituency election winners from one party in a given federal state exceeds its limit, then those with a lower percentage of first votes do not receive mandates. If direct mandates of a party do not exhaust its federal state limit, the remaining mandates are awarded to politicians from the party's federal state list *(Landesliste)*.

To summarize, the Bundestag *regular* size is increased from 598 to 630 seats, and due to restrictions on direct mandates there are no more overhangs and no more extra seats. These measures prevent the Bundestag from growing but the cumulation of rounding errors in the two-tier distribution of Bundestag seats makes the apportionment accuracy of 0.5 seat hardly achievable; for the cumulation of rounding errors see [Mosteller et al. 1967]. Moreover, the reform leaves some constituencies without deputies, making the Bundestag less inclusive regarding local representatives and violating thereby the balance between two concepts of representation in favor of the agent concept at the expense of the descriptive one.

In this paper, we analyze the 2025 Bundestag, which consists of members from seven parties described in Table 1, and focus on the following three points. First, the accuracy of seat allocation can be unconditionally improved by applying discrete optimization techniques instead of the currently used Sainte-Laguë method. This heuristic method dates back to an 1832 proposal by American statesman Daniel Webster and a study of French mathematician André Sainte-Laguë in 1910 [Sainte-Laguë method 2025] — so it is not surprising that modern mathematical tools outperform it.

Second, one can emphasize the descriptive concept of representation weakened by the 2023/24 electoral reform and restore its former balance with the agent concept. For this purpose, the number of federal state party mandates should be calculated directly in one step instead of the two-tier procedure. A positive side effect is that the cumulation of rounding errors in the two-tier procedure can be avoided.

Third, all apportionment problems caused by the 'ideal of one man, one vote' [Balinski and Young 1982] can be completely eliminated by introducing *adjustment vote weights*, resembling power indices in the game theory and its political applications [Shapley and Shubik 1954, Mazurkiewicz and Mercik 2005, Varela and Prado-Dominguez 2012, Holler and Nurmi 2013]. Adjustment vote weights make numerous adjustment seats unnecessary, and even the former 598 regular Bundestag seats turn out to be absolutely sufficient. For this purpose, the members of the most overrepresented party (due to the excess of direct mandates) should have vote weight = 1 and the members of other parties — easily computable vote weights somewhat $> 1$. Thereby, one can constrain the Bundestag growth, retain all direct mandates, restoring the inclusiveness of the Bundestag, and refine the balance of party powers in the Bundestag, bringing it to the exact ratio of votes cast for the parties, which is hardly attainable using integer-valued allocation of seats.

Relaxations of the rule 'one man—one vote' are not that uncommon. For instance, the chairperson of a committee with an even number of members may be given 1.5 votes to avoid a tie. In joint-stock companies, the vote power of each shareholder is proportional to his/her percentage of shares [Edelman et al. 2014], etc. As for the Bundestag, adjustment vote weights do not in the least contradict the established German apportionment rules, since they *already* practice adjustments — in the form of adjustment seats and rounding (= adjusting) fractional numbers of seats obtained from percentages of votes. Besides that, the range of adjustment vote weights is expected to be quite moderate. For the actual 2025 Bundestag with 630 seats and the non-reduced number of 299 direct mandates, the adjustment vote weights would vary in interval $[1.00, 1.09]$, i.e. in the range of 9%. In any case, it seems better to slightly adjust vote weights for some Bundestag members than to leave them without mandates at all. Finally, the technical implementation of adjustment vote weights requires a very simple add-on to the software that supports already existing voting consoles and deputy cards.

To be specific, all the devices suggested in this paper are illustrated using hypothetical redistributions of

Table 1: Parties qualified for the 2025 Bundestag mandates

| Party logo | Party description | Votes in 2021 | | Votes in 2025 | |
|---|---|---|---|---|---|
| | | Number | % | Number | % |
| | CDU/CSU, union of Germany's main conservative parties, Christlich Demokratische Union Deutschlands (Christian Democratic Union of Germany) and Christlich-Soziale Union in Bayern (Christian Social Union of Bavaria). The CDU and CSU were founded in 1945–50 and 1945, respectively, as non-denominational parties and combine conservative, economically liberal and Christian-social positions. The CDU runs in elections in all federal states except for Bavaria, and the CSU runs in elections only there. | 8775471<br>2402827 | 18.896<br>5.174 | 11196374<br>2964028 | 22.551<br>5.970 |
| | AfD, Alternative für Deutschland (Alternative for Germany), was founded in 2013 and focuses, among other things, on restrictive positions in asylum and migration policy. It has been represented in the Bundestag since 2017. The Federal Office for the Protection of the Constitution lists it as a suspected case of right-wing extremist activities. | 4803902 | 10.344 | 10328780 | 20.803 |
| | SPD, Sozial-demokratische Partei Deutschlands (Social Democratic Party of Germany), founded in 1863. The SPD emerged from the workers' movement in 1875. The slogan of social justice is the starting point for many of its positions, for example in the party's labor, social and societal policies. Since the 2021 federal elections it the fourth time in its history that the party has a chancellor. | 11955434 | 25.743 | 8149124 | 16.413 |
| | GRÜNE, BÜNDNIS 90/DIE GRÜNEN (Alliance 90/The Greens). The Greens were founded in 1980 and later joined forces with civil rights movements from the former GDR. They campaign for environmental protection, disarmament, renewable energies and gender equality, among other things. The Greens have been part of the federal government since 2021. | 6852206 | 14.754 | 5762380 | 11.606 |
| | DIE LINKE (The Left) was formed in 2007 through the merger of the PDS, Partei des Demokratischen Sozialismus (Party of Democratic Socialism), and the trade union-oriented WASG, Arbeit und soziale Gerechtigkeit — Die Wahlalternative (Labour and Social Justice — The Electoral Alternative). It is represented in the 2021 Bundestag. The Left advocates disarmament and the expansion of the welfare state, calls for a billionaire and wealth tax and wants to reduce the burden on small and medium incomes. | 2270906 | 4.890 | 4356532 | 8.775 |
| | SSW, Südschleswigscher Wählerverband (South Schleswig Association of Voters), was founded in 1948. It is the political lobby of the Danish minority and the Frisian ethnic group and is therefore exempt from the 5%-hurdle. Its focus is on northern Germany. Since 2021 the SSW is represented by one member of Bundestag. | 55578 | 0.120 | 76138 | 0.153 |
| | Total | 37116324 | 79.920 | 42833356 | 86.271 |

Sources: [Bundeszentrale für politische Bildung 2025, Bundeswahlleiter 2021, Bundeswahlleiterin 2025] seats in the 2025 Bundestag.

In Section 2, 'Official apportionment of the 2025 Bundestag', the so called 'divisor procedure' — a version of the Sainte-Laguë method — is explained step by step, including the reduction in direct mandates prescribed by the 2023/24 electoral reform.

In Section 3, 'Accuracy of distribution of Bundestag mandates between parties', four alternative apportionments of the 2025 Bundestag are computed and analyzed: using the Sainte-Laguë method, the D'Hondt method as well as two optimization models that minimize apportionment inaccuracies measured in seats or in percent of the vote-derived (fractional) quotas.

In Section 4, 'Accuracy of distribution of Bundestag mandates between federal states', the same four methods are applied to distribute the party mandates between federal states.

In Section 5, 'One-tier distribution of Bundestag mandates between federal state party associations', the apportionment procedures operate directly on 81 federal state party associations without first allocating the Bundestag mandates parties and then distributing them between federal states.

In Section 6, 'Adjustment vote weights for making the Bundestag more inclusive', we show that the reduction in direct mandates and leaving thereby some constituencies without representatives in the Bundestag could be avoided if adjustment vote weights were introduced.

In Section 7, 'Concluding discussion', the main findings are recapitulated and put into context.

Section 8, 'Appendix: Apportionment as a mathematical problems', describes four methods for allocating parliamentary seats considered, including mathematical details and computer implementation.

## 2 Official apportionment of the 2025 Bundestag

Let us trace the official allocation of Bundestag seats to the eligible parties step by step.

1. **Distribution of Bundestag mandates between parties.** Table 2 displays a screenshot from the official election report [Bundeswahlleiterin 2025] showing how the Bundestag mandates are distributed among parties using the Saint-Laguë method. There are several equivalent ways to make calculations, and the report follows the so-called *Divisor* approach.

   The calculation starts from the total number of 42,833,356 votes for the eligible parties and 630 Bundestag seats, implying that

   $$\text{Each Bundestag seat must be backed up by} \quad \frac{42,833,356}{630} \approx 67,989.454 \quad \text{votes .}$$

   The resulting figure is called *divisor*. Dividing the votes received by the parties by this divisor and rounding the results to the nearest integer, one obtains the party mandates, which, due to the cumulation of rounding errors, give totally 631 mandates instead of 630. To adjust the total to 630, the divisor is a little increased, in this case from 67,989.454 to 68,100, which gives the apportionment of exactly 630 mandates.[3] For example, the CDU gets 164 Bundestag mandates.

2. **Distribution of party mandates between federal states.** Table 3 displays a screenshot from the election report showing how 164 CDU seats in the Bundestag must be distributed between federal states. Again, the total number of 11,196,374 votes received by the CDU nationwide is divided by 164 seats, implying that

   $$\text{Each CDU Bundestag seat must be backed up by} \quad \frac{11,196,374}{164} \approx 68,270.573 \quad \text{CDU's votes .}$$

[3] Divisor adjustments to correct the cumulated rounding error (see [Mosteller et al. 1967])) result in minor apportionment inaccuracies that are further aggravated in the two-tier distribution of mandates between federal state party associations.

Table 2: Finding the divisor to distribute Bundestag mandates between parties

6 Endgültige Sitzberechnung und Verteilung der Mandate

6.2 Ermittlung der Divisorspanne und des endgültigen Divisors

6.2.1 ... für „6.1.1: ... Verteilung der Gesamtsitzzahl der Parteien"

**1. Berechnungsschritt – Ermittlung des Anfangsdivisors:**

| | |
|---|---|
| Zu berücksichtigende Zweitstimmen: | 42.833.356 |
| Zu verteilende Sitze: | 630 |
| Anfangsdivisor: | 42.833.356 : 630 ≈ 67.989,454 |

| Partei | Zweitstimmen | Divisor | Sitze nach Zweitstimmen | |
|---|---|---|---|---|
| | | | ungerundet | gerundet |
| **SPD** | 8.149.124 | 67.989,454 | 119,858 | **120** |
| **CDU** | 11.196.374 | | 164,678 | **165** |
| **GRÜNE** | 5.762.380 | | 84,754 | **85** |
| **AfD** | 10.328.780 | | 151,917 | **152** |
| **CSU** | 2.964.028 | | 43,595 | **44** |
| **Die Linke** | 4.356.532 | | 64,076 | **64** |
| **SSW** | 76.138 | | 1,119 | **1** |
| **Bundesgebiet** | **42.833.356** | | | **631** |

Bei der Berechnung mit dem Anfangsdivisor sind mehr Sitze auf die Parteien entfallen, als Sitze zu vergeben sind. Deshalb ist der Divisor heraufzusetzen.

**2. Berechnungsschritt – Ermittlung der Divisorspanne und des endgültigen Divisors:**

Der Divisor wird iterativ erhöht. Im ersten Schritt wird aus einer Reihe von Divisorkandidaten ein Divisor ausgewählt, der größer ist als der kleinste Divisorkandidat und kleiner oder gleich dem zweitkleinsten Divisorkandidaten. Dieser Schritt wird so lange wiederholt, bis sich bei der Berechnung mit dem ausgewählten Divisor die vorgegebene Sitzzahl ergibt. Bei der Division negative Sitzzahlen bleiben unberücksichtigt.

| Partei | Ermittlung der Divisorkandidaten | | | | | Berechnung der Sitze | |
|---|---|---|---|---|---|---|---|
| | Zweitstimmen | Division mit Sitzzahl aus vorherigem Schritt – 0,5 | = Divisor-kandidat 1 | Division mit Sitzzahl aus vorherigem Schritt – 1,5 | = Divisor-kandidat 2 | Ausgewählter Divisor | Sitze gerundet |
| | **1. Iterationsschritt** | | | | | | |
| **SPD** | 8.149.124 | 119,5 | 68.193,506 | 118,5 | 68.768,979 | Divisor- | **120** |
| **CDU** | 11.196.374 | 164,5 | 68.063,064 | 163,5 | 68.479,352 | spanne: | **164** |
| **GRÜNE** | 5.762.380 | 84,5 | 68.193,846 | 83,5 | 69.010,539 | > 68.063,064 | **85** |
| **AfD** | 10.328.780 | 151,5 | 68.176,766 | 150,5 | 68.629,767 | und | **152** |
| **CSU** | 2.964.028 | 43,5 | 68.138,575 | 42,5 | 69.741,835 | ≤ 68.138,575 | **44** |
| **Die Linke** | 4.356.532 | 63,5 | 68.606,803 | 62,5 | 69.704,512 | | **64** |
| **SSW** | 76.138 | 0,5 | 152.276,000 | | – | Ausgewählter Divisor: 68.100 | **1** |
| **Bundesgebiet** | **42.833.356** | | | | | | **630** |

| | |
|---|---|
| Mögliche Divisorspanne: | > 68.063,064 und ≤ 68.138,575 |
| Ausgewählter Divisor: | **68.100** |

Source: [Bundeswahlleiterin 2025, Screenshot of p. 411]

Table 3: Finding the divisor to distribute the CDU mandates between federal states

**CDU**

**1. Berechnungsschritt – Ermittlung des Anfangsdivisors:**

Zu berücksichtigende Zweitstimmen: 11.196.374
Zu verteilende Sitze: 164
Anfangsdivisor: 11.196.374 : 164 = 68.270,573

| Bundesland | Zweitstimmen | Divisor | Sitze | |
|---|---|---|---|---|
| | | | ungerundet | gerundet |
| Schleswig-Holstein | 518.424 | | 7,593 | 8 |
| Mecklenburg-Vorpommern | 181.956 | | 2,665 | 3 |
| Hamburg | 216.935 | | 3,177 | 3 |
| Niedersachsen | 1.410.418 | | 20,659 | 21 |
| Bremen | 71.573 | | 1,048 | 1 |
| Brandenburg | 298.048 | | 4,365 | 4 |
| Sachsen-Anhalt | 256.538 | | 3,757 | 4 |
| Berlin | 356.099 | 68.270,573 | 5,215 | 5 |
| Nordrhein-Westfalen | 3.170.627 | | 46,442 | 46 |
| Sachsen | 507.247 | | 7,429 | 7 |
| Hessen | 1.033.842 | | 15,143 | 15 |
| Thüringen | 246.065 | | 3,604 | 4 |
| Rheinland-Pfalz | 760.623 | | 11,141 | 11 |
| Baden-Württemberg | 2.006.866 | | 29,395 | 29 |
| Saarland | 161.113 | | 2,359 | 2 |
| **Zusammen** | **11.196.374** | | | **163** |

Bei der Berechnung mit dem Anfangsdivisor sind weniger Sitze auf die Landeslisten der Parteien entfallen, als Sitze zu vergeben sind. Deshalb ist der Divisor herabzusetzen.

**2. Berechnungsschritt – Ermittlung der Divisorspanne und des endgültigen Divisors:**

Der Divisor wird iterativ herabgesetzt. Im ersten Schritt wird aus einer Reihe von Divisorkandidaten ein Divisor ausgewählt, der größer ist als der zweitgrößte Divisorkandidat und kleiner oder gleich dem größten Divisorkandidaten. Dieser Schritt wird so lange wiederholt, bis sich bei der Berechnung mit dem ausgewählten Divisor die vorgegebene Sitzzahl ergibt.

| Bundesland | Ermittlung der Divisorkandidaten | | | | | Berechnung der Sitze | |
|---|---|---|---|---|---|---|---|
| | Zweitstimmen | Division mit Sitzzahl aus vorherigem Schritt + 0,5 | = Divisor-kandidat 1 | Division mit Sitzzahl aus vorherigem Schritt + 1,5 | = Divisor-kandidat 2 | Ausgewählter Divisor | Sitze gerundet |
| | **1. Iterationsschritt** | | | | | | |
| Schleswig-Holstein | 518.424 | 8,5 | 60.991,059 | 9,5 | 54.570,947 | | 8 |
| Mecklenburg-Vorpommern | 181.956 | 3,5 | 51.987,429 | 4,5 | 40.434,667 | | 3 |
| Hamburg | 216.935 | 3,5 | 61.981,429 | 4,5 | 48.207,778 | | 3 |
| Niedersachsen | 1.410.418 | 21,5 | 65.600,837 | 22,5 | 62.685,244 | Divisor- | 21 |
| Bremen | 71.573 | 1,5 | 47.715,333 | 2,5 | 28.629,200 | spanne: | 1 |
| Brandenburg | 298.048 | 4,5 | 66.232,889 | 5,5 | 54.190,545 | > 68.029,356 | 4 |
| Sachsen-Anhalt | 256.538 | 4,5 | 57.008,444 | 5,5 | 46.643,273 | und | 4 |
| Berlin | 356.099 | 5,5 | 64.745,273 | 6,5 | 54.784,462 | ≤ 68.185,527 | 5 |
| Nordrhein-Westfalen | 3.170.627 | 46,5 | 68.185,527 | 47,5 | 66.750,042 | | 47 |
| Sachsen | 507.247 | 7,5 | 67.632,933 | 8,5 | 59.676,118 | Ausgewählter | 7 |
| Hessen | 1.033.842 | 15,5 | 66.699,484 | 16,5 | 62.657,091 | Divisor: | 15 |
| Thüringen | 246.065 | 4,5 | 54.681,111 | 5,5 | 44.739,091 | 68.100 | 4 |
| Rheinland-Pfalz | 760.623 | 11,5 | 66.141,130 | 12,5 | 60.849,840 | | 11 |
| Baden-Württemberg | 2.006.866 | 29,5 | 68.029,356 | 30,5 | 65.798,885 | | 29 |
| Saarland | 161.113 | 2,5 | 64.445,200 | 3,5 | 46.032,286 | | 2 |
| **Zusammen** | **11.196.374** | | | | | | **164** |

Mögliche Divisorspanne: > 68.029,356 und ≤ 68.185,527
Ausgewählter Divisor: **68.100**

Source: [Bundeswahlleiterin 2025, Screenshot of pp. 414–415]

Table 4: Reduction in a direct mandate in the federal state of Schleswig-Holtstein

4 Endgültig Gewählte

4.1 Kreiswahlvorschläge

| Land / Wahlkreis | | Name, Vornamen | Partei | Erststimmen | | Erstunterlegen | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | Partei | Stimmenabstand | |
| | | | | Anzahl | % | | Anzahl | %-Pkt. |
| Schleswig-Holstein | | | | | | | | |
| 001 | Flensburg – Schleswig | [1] | CDU | 50.838 | 26,5 | GRÜNE | 7.540 | 3,9 |
| 002 | Nordfriesland – Dithmarschen Nord | Bodin, Leif Erik | CDU | 49.866 | 32,7 | SPD | 18.697 | 12,2 |
| 003 | Steinburg – Dithmarschen Süd | Helfrich, Mark | CDU | 50.887 | 35,0 | SPD | 20.185 | 13,9 |
| 004 | Rendsburg-Eckernförde | Dr. Wadephul, Johann David Walter Rudolf | CDU | 56.163 | 32,8 | SPD | 19.897 | 11,6 |
| 005 | Kiel | Amtsberg, Luise | GRÜNE | 43.281 | 26,0 | SPD | 6.591 | 4,0 |
| 006 | Plön – Neumünster | Carstensen, Sandra | CDU | 46.370 | 32,7 | SPD | 12.393 | 8,7 |
| 007 | Pinneberg | Kölbl, Daniel | CDU | 63.232 | 31,8 | SPD | 13.518 | 6,8 |
| 008 | Segeberg – Stormarn-Mitte | Bernstein, Melanie | CDU | 67.880 | 32,3 | SPD | 16.443 | 7,8 |
| 009 | Ostholstein – Stormarn-Nord | Schmidt, Sebastian | CDU | 52.024 | 34,8 | SPD | 13.954 | 9,3 |
| 010 | Herzogtum Lauenburg – Stormarn-Süd | Schmidt, Henri | CDU | 67.642 | 32,7 | SPD | 19.367 | 9,4 |
| 011 | Lübeck | Klüssendorf, Tim | SPD | 39.809 | 28,1 | CDU | 6.114 | 4,3 |

1 Wahlkreis 001 (SH) bleibt wegen ungenügender Zweitstimmendeckung unbesetzt.

Source: [Bundeswahlleiterin 2025, Screenshot of p. 333]

Dividing the votes received by the CDU in federal states by this divisor and rounding the results we obtain that the sum of the rounded numbers is equal to $163 < 164$. Reducing a little this divisor down to 68,100, we get the correct total of 164 CDU seats, which gives the final distribution of the CDU mandates among the federal states.

For other parties, the distribution of seats between federal states is calculated similarly; for details see [Bundeswahlleiterin 2025, pp. 412–423].

3. **Reduction in direct mandates.** As follows from the bottom section of Table 3, the number of CDU mandates in the federal state of Schleswig-Holstein is limited to eight. On the other hand, CDU won the first vote elections in its nine constituencies; see Table 4. Therefore, the CDU 'weakest' constituency election winner with the least 26.5% of the CDU first votes, the one in the Constituency 001, Flensburg-Schleswig, is not awarded with a direct mandate — the corresponding table cell is blank with an explanation in the footnote to the table: 'Constituency 001 (SH) remains vacant because of insufficient coverage by the second vote'. Since eight direct mandates exhaust the Schleswig-Holstein CDU limit, no other politician from the CDU federal state list gets a Bundestag seat. This means that Constituency 001 Flensburg-Schleswig delegates no representative to the Bundestag, and the whole federal state of Schleswig-Holstein, missing a deputy from one of its constituencies, is underrepresented.

   According to [Bundeswahlleiterin 2025, pp. 333–340], most CDU constituency election winners are awarded with direct mandates, but not all. For example, the federal state of Hessen is five CDU direct mandates short, and Baden-Württemberg even six [Bundeswahlleiterin 2025, pp. 337, 339].

   For other parties, the procedure is the same.

4. **Final distribution of Bundestag mandates among federal states and parties.** The party mandates, direct and awarded to politicians from the federal state lists are displayed in Table 5. For ex-

Table 5: Final allocation of Bundestag mandates to parties and federal states

3 Endgültige Bundestagsmandate nach Parteien und Länder

| Partei | Sitze | Bund | SH | MV | HH | NI | HB | BB | ST | BE | NW | SN | HE | TH | RP | BY | BW | SL |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Insgesamt** | **Insgesamt** | **630** | **25** | **13** | **13** | **65** | **5** | **21** | **16** | **24** | **136** | **30** | **45** | **18** | **31** | **101** | **79** | **8** |
| | Diff. zu 2021 | -105 | -3 | -3 | -3 | -9 | – | -4 | -2 | -1 | -20 | -8 | -6 | -1 | -5 | -16 | -23 | -1 |
| | Wahlkreis | **276** | 10 | 5 | 6 | 30 | 1 | 9 | 7 | 12 | 64 | 15 | 17 | 8 | 12 | 44 | 32 | 4 |
| | Landesliste | **354** | 15 | 8 | 7 | 35 | 4 | 12 | 9 | 12 | 72 | 15 | 28 | 10 | 19 | 57 | 47 | 4 |
| davon: | | | | | | | | | | | | | | | | | | |
| **SPD** | **Zusammen** | **120** | **5** | **2** | **3** | **17** | **1** | **4** | **2** | **4** | **31** | **3** | **10** | **2** | **7** | **14** | **13** | **2** |
| | Diff. zu 2021 | -86 | -3 | -4 | -2 | -9 | -1 | -6 | -3 | -2 | -18 | -5 | -5 | -3 | -5 | -9 | -9 | -2 |
| | Wahlkreis | **44** | 1 | – | 3 | 15 | 1 | 1 | – | 1 | 17 | – | 2 | – | 1 | – | – | 2 |
| | Landesliste | **76** | 4 | 2 | – | 2 | – | 3 | 2 | 3 | 14 | 3 | 8 | 2 | 6 | 14 | 13 | – |
| **CDU** | **Zusammen** | **164** | **8** | **3** | **3** | **21** | **1** | **4** | **4** | **5** | **47** | **7** | **15** | **4** | **11** | **–** | **29** | **2** |
| | Diff. zu 2021 | +12 | +2 | – | – | +3 | – | – | – | – | +5 | – | +3 | +1 | +2 | – | -4 | – |
| | Wahlkreis | **128** | 8 | – | 1 | 15 | – | – | – | 3 | 44 | – | 15 | – | 11 | – | 29 | 2 |
| | Landesliste | **36** | – | 3 | 2 | 6 | 1 | 4 | 4 | 2 | 3 | 7 | – | 4 | – | – | – | – |
| **GRÜNE** | **Zusammen** | **85** | **4** | **1** | **3** | **8** | **1** | **2** | **1** | **5** | **19** | **2** | **7** | **1** | **4** | **14** | **12** | **1** |
| | Diff. zu 2021 | -33 | -2 | – | -1 | -5 | – | – | – | -1 | -9 | -2 | -2 | – | -1 | -5 | -6 | +1 |
| | Wahlkreis | **12** | 1 | – | 2 | – | – | – | – | 3 | 3 | – | – | – | – | – | 3 | – |
| | Landesliste | **73** | 3 | 1 | 1 | 8 | 1 | 2 | 1 | 2 | 16 | 2 | 7 | 1 | 4 | 14 | 9 | 1 |
| **AfD** | **Zusammen** | **152** | **5** | **5** | **2** | **13** | **1** | **8** | **7** | **4** | **26** | **14** | **9** | **8** | **7** | **22** | **19** | **2** |
| | Diff. zu 2021 | +69 | +3 | +2 | +1 | +7 | +1 | +3 | +3 | +1 | +14 | +4 | +4 | +3 | +3 | +10 | +9 | +1 |
| | Wahlkreis | **42** | – | 5 | – | – | – | 8 | 7 | 1 | – | 14 | – | 7 | – | – | – | – |
| | Landesliste | **110** | 5 | – | 2 | 13 | 1 | – | – | 3 | 26 | – | 9 | 1 | 7 | 22 | 19 | 2 |
| **CSU** | **Zusammen** | **44** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **44** | **–** | **–** |
| | Diff. zu 2021 | -1 | – | – | – | – | – | – | – | – | – | – | – | – | – | -1 | – | – |
| | Wahlkreis | **44** | – | – | – | – | – | – | – | – | – | – | – | – | – | 44 | – | – |
| | Landesliste | **–** | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |
| **Die Linke** | **Zusammen** | **64** | **2** | **2** | **2** | **6** | **1** | **3** | **2** | **6** | **13** | **4** | **4** | **3** | **2** | **7** | **6** | **1** |
| | Diff. zu 2021 | +25 | +1 | – | +1 | +3 | +1 | +1 | – | +3 | +7 | – | +1 | – | +1 | +3 | +3 | – |
| | Wahlkreis | **6** | – | – | – | – | – | – | – | 4 | – | 1 | – | 1 | – | – | – | – |
| | Landesliste | **58** | 2 | 2 | 2 | 6 | 1 | 3 | 2 | 2 | 13 | 3 | 4 | 2 | 2 | 7 | 6 | 1 |
| **SSW** | **Zusammen** | **1** | **1** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** | **–** |
| | Diff. zu 2021 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |
| | Wahlkreis | **–** | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |
| | Landesliste | **1** | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

Source: [Bundeswahlleiterin 2025, Screenshot of p. 331]. The federal state names in the table heading are abbreviated; for the full names see Table 3.

ample, the CDU section shows the federal state limits in Row ‘Zusammen’ (= Together) highlighted in bold. The number of awarded direct mandates is shown in Row ‘Wahlkreis’ (= Constituency). If the direct mandates do not exhaust the federal state limit then the number of remaining mandates is indicated in Row ‘Landesliste’ (= Federal state list).

The upper section of the table shows the allocation of all Bundestag mandates to federal states (disregarding the parties). In particular, the total reduction in direct mandates is seen in their number of $276 < 299$ constituencies, illustrating the restrictive effect of the 2023/24 electoral reform.

Thus, 23 out of 299 constituencies ($\approx 7.7\%$) delegate no local representative to the Bundestag, and the parties complete their Bundestag quotas with politicians on their own choice. This means a clear bias toward the agent concept of representation at the expense of the descriptive one.

## 3 Accuracy of distribution of Bundestag mandates between parties

To avoid misinterpretations, by Bundestag quota $q_i,\ 0 < q_i < 1$ of party $i$ we always understand its share of the votes cast for the parties eligible for Bundestag seats; if it is given in percent then $q_i$ is multiplied by 100. For example, the CDU’s 11196374 votes in Table 1, being reduced to 42833356 votes cast for the seven Bundestag paries (see at the bottom of the table), gives the CDU Bundestag quota

$$q_{\text{CDU}} = \frac{11196374}{42833356} \approx 0.26139 \quad (= 26.139\%)\ .$$

Thus, a quota is a vote-determined fractional constant. Its conversion into the integer number of Bundestag seats is not unique, depending on fairness criteria and apportionment methods. In this paper, we consider four methods described in Section 8, ‘Appendix: Apportionment as a mathematical problems’:

**Saint-Laguë method,** used to distribute Bundestag seats since 2008 [Sainte-Laguë-Verfahren 2025],

**D’Hondt method** used to distribute Bundestag seats from 1949 to 1985 and still used in federal state parliaments *(Landesparlamente)* [D’Hondt-Verfahren 2025],[4]

**Absolute optimization** that minimizes *absolute* apportionment errors measured in fractional number of seats, and

**Relative optimization** that minimizes *relative* apportionment errors measured in percentage of the quota — to reflect the different importance of one seat for large and small parliamentary factions; as we see later, Relative optimization reduces the inequality between Bundestag members if adjustment vote weights are applied.

Four sections of Table 6 characterize distributions of Bundestag seats calculated using four methods listed above. The columns in the first table section are numbered from 10 to 19, in the second section — from 20 to 29, etc., with the first decimal indicating the section number (method) and the second decimal indicating the same content of the column in all sections. At the moment we focus on the following columns:

$*0$ $\quad \mathbf{q} * 100, \quad \mathbf{q} = \{q_i,\ i = \text{CDU},\ \ldots,\ \text{SSW}\}$ — the vector of party Bundestag quotas given in %,

$*3$ $\quad \mathbf{x} = \{x_i,\ \ i = \text{CDU},\ \ldots,\ \text{SSW}\}$ — the apportionment vector with integer numbers of Bundestag seats allocated to the eligible parties,

[4] We do not consider the Hamilton/Haare/Niemeyer method (also known as Quota method), with which the Bundestag seats were distributed from 1985 to 2008 [Hare/Niemeyer-Verfahren 2025]. This method is less practiced than the Saint-Laguë and D’Hondt methods because of paradoxical outcomes in certain situations [Quota method 2025].

Table 6: Distribution of 630 Bundestag seats between 7 parties

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Using **Sainte-Laguë** method | | | | | | | | | | |
| | Proportion of votes (quota), in % | Direct mandates | Adjustment seats | | Seats (apportionment) | Number of seats according to the quota | Absolute deviation from the quota, in number of seats | Relative deviation from the quota, in % of the quota | Adjustment vote weight | Party's votes (Seats × Vote weight) | | Share of power (Seats × Vote weight), in % |
| | $\mathbf{q}*100$ | $\mathbf{d}$ | $\mathbf{a}$ | | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'*\mathbf{w}}*100$ |
| | 10 | 11 | 12 | | 13 | 14 | 15 | 16 | 17 | 18 | | 19 |
| CDU | 26.139 | – | 164 | = | 164 | 164.678 | $\boxed{-0.678}$ | −0.412 | 1.013 | $\boxed{166.206}$ | → | 26.139 |
| AfD | 24.114 | – | 152 | = | 152 | 151.917 | 0.083 | 0.054 | 1.009 | 153.327 | → | 24.114 |
| SPD | 19.025 | – | 120 | = | 120 | 119.859 | 0.141 | 0.118 | 1.008 | 120.971 | → | 19.025 |
| GRÜNE | 13.453 | – | 85 | = | 85 | 84.754 | 0.246 | 0.290 | 1.006 | 85.541 | → | 13.453 |
| LINKE | 10.171 | – | 64 | = | 64 | 64.077 | −0.077 | −0.120 | 1.010 | 64.671 | → | 10.171 |
| CSU | 6.920 | – | 44 | = | 44 | 43.595 | $\boxed{0.405}$ | $\boxed{0.928}$ | $\boxed{1.000}$ | 44.000 | → | 6.920 |
| SSW | 0.178 | – | 1 | = | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.130}$ | $\boxed{1.130}$ | → | 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | – | 630 | = | 630 | 630.000 | $\boxed{1.083}$ | $\boxed{11.630}$ | $\boxed{0.130}$ | $\boxed{165.076}$ | → | 100.000 |
| | | Using **D'Hondt** method | | | | | | | | | | |
| | $\mathbf{q}*100$ | $\mathbf{d}$ | $\mathbf{a}$ | | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}*100$ |
| | 20 | 21 | 22 | | 23 | 24 | 25 | 26 | 27 | 28 | | 29 |
| CDU | 26.139 | – | 165 | = | 165 | 164.678 | $\boxed{0.322}$ | 0.195 | 1.001 | $\boxed{165.156}$ | → | 26.139 |
| AfD | 24.114 | – | 152 | = | 152 | 151.917 | 0.083 | 0.054 | 1.002 | 152.358 | → | 24.114 |
| SPD | 19.025 | – | 120 | = | 120 | 119.859 | 0.141 | 0.118 | 1.002 | 120.207 | → | 19.025 |
| GRÜNE | 13.453 | – | 85 | = | 85 | 84.754 | 0.246 | $\boxed{0.290}$ | $\boxed{1.000}$ | 85.000 | → | 13.453 |
| LINKE | 10.171 | – | 64 | = | 64 | 64.077 | −0.077 | −0.120 | 1.004 | 64.263 | → | 10.171 |
| CSU | 6.920 | – | 43 | = | 43 | 43.595 | $\boxed{-0.595}$ | −1.366 | 1.017 | 43.722 | → | 6.920 |
| SSW | 0.178 | – | 1 | = | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.123}$ | $\boxed{1.123}$ | → | 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | – | 630 | = | 630 | 630.000 | $\boxed{0.917}$ | $\boxed{10.993}$ | $\boxed{0.123}$ | $\boxed{164.033}$ | → | 100.000 |
| | | Using **Absolute** optimization | | | | | | | | | | |
| | $\mathbf{q}*100$ | $\mathbf{d}$ | $\mathbf{a}$ | | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}*100$ |
| | 30 | 31 | 32 | | 33 | 34 | 35 | 36 | 37 | 38 | | 39 |
| CDU | 26.139 | – | 165 | = | 165 | 164.678 | $\boxed{0.322}$ | 0.195 | 1.001 | $\boxed{165.156}$ | → | 26.139 |
| AfD | 24.114 | – | 152 | = | 152 | 151.917 | 0.083 | 0.054 | 1.002 | 152.358 | → | 24.114 |
| SPD | 19.025 | – | 120 | = | 120 | 119.859 | 0.141 | 0.118 | 1.002 | 120.207 | → | 19.025 |
| GRÜNE | 13.453 | – | 85 | = | 85 | 84.754 | 0.246 | $\boxed{0.290}$ | $\boxed{1.000}$ | 85.000 | → | 13.453 |
| LINKE | 10.171 | – | 64 | = | 64 | 64.077 | −0.077 | −0.120 | 1.004 | 64.263 | → | 10.171 |
| CSU | 6.920 | – | 43 | = | 43 | 43.595 | $\boxed{-0.595}$ | −1.366 | 1.017 | 43.722 | → | 6.920 |
| SSW | 0.178 | – | 1 | = | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.123}$ | $\boxed{1.123}$ | → | 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | – | 630 | = | 630 | 630.000 | $\boxed{0.917}$ | $\boxed{10.993}$ | $\boxed{0.123}$ | $\boxed{164.033}$ | → | 100.000 |
| | | Using **Relative** optimization | | | | | | | | | | |
| | $\mathbf{q}*100$ | $\mathbf{d}$ | $\mathbf{a}$ | | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}*100$ |
| | 40 | 41 | 42 | | 43 | 44 | 45 | 46 | 47 | 48 | | 49 |
| CDU | 26.139 | – | 165 | = | 165 | 164.678 | $\boxed{0.322}$ | 0.195 | 1.001 | $\boxed{165.156}$ | → | 26.139 |
| AfD | 24.114 | – | 152 | = | 152 | 151.917 | 0.083 | 0.054 | 1.002 | 152.358 | → | 24.114 |
| SPD | 19.025 | – | 120 | = | 120 | 119.859 | 0.141 | 0.118 | 1.002 | 120.207 | → | 19.025 |
| GRÜNE | 13.453 | – | 85 | = | 85 | 84.754 | 0.246 | $\boxed{0.290}$ | $\boxed{1.000}$ | 85.000 | → | 13.453 |
| LINKE | 10.171 | – | 64 | = | 64 | 64.077 | −0.077 | −0.120 | 1.004 | 64.263 | → | 10.171 |
| CSU | 6.920 | – | 43 | = | 43 | 43.595 | $\boxed{-0.595}$ | −1.366 | 1.017 | 43.722 | → | 6.920 |
| SSW | 0.178 | – | 1 | = | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.123}$ | $\boxed{1.123}$ | → | 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | – | 630 | = | 630 | 630.000 | $\boxed{0.917}$ | $\boxed{10.993}$ | $\boxed{0.123}$ | $\boxed{164.033}$ | → | 100.000 |

$*4$ $\mathbf{q} * S$ — the vector of party quotas expressed in the (fractional) number of Bundestag seats; here, the number of Bundestag seats $S = 630$.

$*5$ $\mathbf{x} - S * \mathbf{q}$ — the vector of absolute apportionment errors, given in the number of seats. It is the difference between columns * 3 and * 4, For the Sainte-Laguë method (Column 15),

$$\text{CDU absolute apportionment error } = 164 - 164.678 = -0.678 \ ,$$

meaning that the CDU is underrepresented by 0.678 seat.

$*6$ $\left(\frac{\mathbf{x}}{S} - \mathbf{q}\right) ./\mathbf{q} * 100$ — the vector of relative apportionment errors given in percent, that is, the deviation from the quota relative to the quota size.[5] For the Sainte-Laguë method (Column 16),

$$\text{CDU relative apportionment error } \approx \frac{164/630 - 0.26139}{0.26139} * 100 \approx -0.41\% \ ,$$

meaning that the CDU is underrepresented by 0.412% relative to its Bundestag quota.

In Table 6, the maxima and minima of the columns are framed, and the bottom row indicates either the total of the columns, or their range shown then in frames.

The official Bundestag apportionment using the Sainte-Laguë method in the upper table section (Columns 10–19) differs from that in other other sections (Columns 20–49) in one seat of CDU and CSU, being less accurate than that obtained using other methods. Indeed, the maximum negative absolute apportionment error of $-0.678$ seat in Column 15 (inherent in the CDU) is greater than that of $-0.595$ in Columns 25, 35 and 45 (in this case inherent in the CSU). Since the tolerated rounding error is equal to 0.5 seat, the deviation from this limit $0.678 - 0.5 = 0.178$ is almost two times (!) greater than that $0.595 - 0.5 = 0.095$, also implying smaller relative apportionment errors (Columns * 6) and their smaller range shown in the last rows of Columns $*5$ and $*6$ in frames.

All of these mean that the 2023/24 electoral reform does not fulfill its purpose — to limit the Bundestag size, ensuring a seat distribution accuracy of 0.5 seat. The officially accepted Sainte-Laguë method does not even minimize the apportionment error, because other methods outperform it. It should be however noted that the superiority of the D'Hondt method over the Sainte-Laguë method here is rather occasional. Both methods are not optimal but heuristic and, as we see later, both fail to meet optimization criteria under certain circumstances. This is not the case of the rigorous optimization methods designed to minimize apportionment inaccuracies in all situations. We conclude that the 2023/24 electoral reform relies on an outdated method of apportionment. Its accuracy and, respectively, its justice and fairness, can be improved by using modern mathematical tools.

## 4 Accuracy of distribution of Bundestag mandates between federal states

As described in Paragraph 2 of Section 2, the Bundestag mandates of every party are distributed between its federal state associations in proportion to the votes cast for the party in the federal states. The corresponding Bundestag sub-quotas for party federal state associations are derived from the election results by party and federal state shown in Table 7. These sub-quotas serve reducing in direct mandates that can be awarded or not awarded to the constituency election winners in Table 8.

Officially, the party Bundestag mandates are distributed between federal states using the same Sainte-Laguë method. Since we deal with four apportionment methods, we apply them to this task as well in the following way: if the Bundestag seats were distributed between parties using the Sainte-Laguë method then we distribute them between federal states also using the Sainte-Laguë method; if the D'Hondt method was used, then we also apply the D'Hondt method, etc. Table 9 shows the four distributions of the CDU

[5] The operation ./ means element-by-element division of vectors; for example $\{10,20\}./\{5,2\} = \{2,10\}$.

Table 7: Number of second votes cast for eligible parties in federal states

| Federal state | Bundestag parties | | | | | |
|---|---|---|---|---|---|---|
| | CDU | | AfD | | SPD | |
| | Number | % | Number | % | Number | % |
| SH Schleswig-Holstein | 518424 | 4.630 | 306165 | 2.964 | 352546 | 4.326 |
| MV Mecklenburg-Vorpommern | 181956 | 1.625 | 357361 | 3.460 | 126687 | 1.555 |
| HH Hamburg | 216935 | 1.938 | 113608 | 1.100 | 237740 | 2.917 |
| NI Niedersachsen | 1410418 | 12.597 | 894540 | 8.661 | 1153523 | 14.155 |
| HB Bremen | 71573 | 0.639 | 52496 | 0.508 | 80604 | 0.989 |
| BB Brandenburg | 298048 | 2.662 | 535275 | 5.182 | 244010 | 2.994 |
| ST Sachsen-Anhalt | 256538 | 2.291 | 496110 | 4.803 | 146535 | 1.798 |
| BE Berlin | 356099 | 3.180 | 296990 | 2.875 | 295182 | 3.622 |
| NW Nordrhein-Westfalen | 3170627 | 28.318 | 1770379 | 17.140 | 2108434 | 25.873 |
| SN Sachsen | 507247 | 4.530 | 958401 | 9.279 | 217144 | 2.665 |
| HE Hessen | 1033842 | 9.234 | 636778 | 6.165 | 657510 | 8.068 |
| TH Thüringen | 246065 | 2.198 | 510527 | 4.943 | 115915 | 1.422 |
| RP Rheinland-Pfalz | 760623 | 6.793 | 498695 | 4.828 | 462705 | 5.678 |
| BY Bayern | – | – | 1515731 | 14.675 | 920675 | 11.298 |
| BW Baden-Württemberg | 2006866 | 17.924 | 1256430 | 12.164 | 898778 | 11.029 |
| SL Saarland | 161113 | 1.439 | 129294 | 1.252 | 131136 | 1.609 |
| Total | 11196374 | 100.000 | 10328780 | 100.000 | 8149124 | 100.000 |

| Federal state | Bundestag parties | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | GRÜNE | | LINKE | | CSU | | SSW | |
| | Number | % | Number | % | Number | % | Number | % |
| SH Schleswig-Holstein | 279923 | 4.858 | 146428 | 3.361 | – | – | 76138 | 100.000 |
| MV Mecklenburg-Vorpommern | 54719 | 0.950 | 123059 | 2.825 | – | – | – | – |
| HH Hamburg | 201713 | 3.501 | 151115 | 3.469 | – | – | – | – |
| NI Niedersachsen | 576845 | 10.011 | 405519 | 9.308 | – | – | – | – |
| HB Bremen | 54280 | 0.942 | 51461 | 1.181 | – | – | – | – |
| BB Brandenburg | 108598 | 1.885 | 176224 | 4.045 | – | – | – | – |
| ST Sachsen-Anhalt | 59077 | 1.025 | 143807 | 3.301 | – | – | – | – |
| BE Berlin | 328035 | 5.693 | 387222 | 8.888 | – | – | – | – |
| NW Nordrhein-Westfalen | 1300901 | 22.576 | 877123 | 20.134 | – | – | – | – |
| SN Sachsen | 167269 | 2.903 | 290462 | 6.667 | – | – | – | – |
| HE Hessen | 451510 | 7.835 | 311058 | 7.140 | – | – | – | – |
| TH Thüringen | 56097 | 0.974 | 200688 | 4.607 | – | – | – | – |
| RP Rheinland-Pfalz | 256869 | 4.458 | 161867 | 3.716 | – | – | – | – |
| BY Bayern | 957435 | 16.615 | 456935 | 10.489 | 2964028 | 100.000 | – | – |
| BW Baden-Württemberg | 865738 | 15.024 | 429484 | 9.858 | – | – | – | – |
| SL Saarland | 43371 | 0.753 | 44080 | 1.012 | – | – | – | – |
| Total | 5762380 | 100.000 | 4356532 | 100.000 | 2964028 | 100.000 | 76138 | 100.000 |

Source: [Bundeswahlleiterin 2025, collected from Tables 1.2.1–1.2.16 in pp. 9–24]

Table 8: Number of constituency election winners (direct mandate candidates) by federal state and party; for federal state name abbreviations see Table 7.

| | | Federal states | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Total | SH | MV | HH | NI | HB | BB | ST | BE | NW | SN | HE | TH | RP | BY | BW | SL |
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| CDU | 143 | 9 | – | 1 | 15 | – | – | – | 3 | 44 | – | 20 | – | 14 | – | 35 | 2 |
| AfD | 46 | – | 6 | – | – | – | 9 | 8 | 1 | – | 15 | – | 7 | – | – | – | – |
| SPD | 45 | 1 | – | 3 | 15 | 2 | 1 | – | 1 | 17 | – | 2 | – | 1 | – | – | 2 |
| GRÜNE | 12 | 1 | – | 2 | – | – | – | – | 3 | 3 | – | – | – | – | – | 3 | – |
| LINKE | 6 | – | – | – | – | – | – | – | 4 | – | 1 | – | 1 | – | – | – | – |
| CSU | 47 | – | – | – | – | – | – | – | – | – | – | – | – | – | 47 | – | – |
| SSW | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

Source: [Bundeswahlleiterin 2025, collected from Tables 4.1 in pp. 333–340]

Table 9: Distribution of the CDU Bundestag mandates between federal states. The federal state name abbreviations are explained in Table 7.

| | | Using **Sainte-Laguë** method | | | |
|---|---|---|---|---|---|
| | Proportion of votes (quota), in % | Seats (apportionment) | Number of seats according to the quota | Absolute deviation from the quota, in number of seats | Relative deviation from the quota, in % of the quota |
| | $\mathbf{q}*100$ | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})\,./\mathbf{q}*100$ |
| | 10 | 13 | 14 | 15 | 16 |
| CDU in SH | 4.630 | 8 | 7.594 | 0.406 | 5.351 |
| CDU in MV | 1.625 | 3 | 2.665 | 0.335 | $\boxed{12.561}$ |
| CDU in HH | 1.938 | 3 | 3.178 | −0.178 | −5.588 |
| CDU in NI | 12.597 | 21 | 20.659 | 0.341 | 1.649 |
| CDU in HB | 0.639 | 1 | 1.048 | −0.048 | −4.614 |
| CDU in BB | 2.662 | 4 | 4.366 | −0.366 | −8.376 |
| CDU in ST | 2.291 | 4 | 3.758 | 0.242 | 6.449 |
| CDU in BE | 3.180 | 5 | 5.216 | −0.216 | −4.141 |
| CDU in NW | 28.318 | 47 | 46.442 | $\boxed{0.558}$ | 1.201 |
| CDU in SN | 4.530 | 7 | 7.430 | $\boxed{-0.430}$ | −5.787 |
| CDU in HE | 9.234 | 15 | 15.143 | −0.143 | −0.946 |
| CDU in TH | 2.198 | 4 | 3.604 | 0.396 | 10.980 |
| CDU in RP | 6.793 | 11 | 11.141 | −0.141 | −1.268 |
| CDU in BY | – | – | – | – | – |
| CDU in BW | 17.924 | 29 | 29.396 | −0.396 | −1.346 |
| CDU in SL | 1.439 | 2 | 2.360 | −0.360 | $\boxed{-15.251}$ |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 164 | 164.000 | $\boxed{0.988}$ | $\boxed{27.812}$ |

| | | Using **D'Hondt** method | | | |
|---|---|---|---|---|---|
| | Proportion of votes (quota), in % | Seats (apportionment) | Number of seats according to the quota | Absolute deviation from the quota, in number of seats | Relative deviation from the quota, in % of the quota |
| | $\mathbf{q}*100$ | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})\,./\mathbf{q}*100$ |
| | 20 | 23 | 24 | 25 | 26 |
| CDU in SH | 4.630 | 8 | 7.640 | 0.360 | 4.712 |
| CDU in MV | 1.625 | 2 | 2.681 | −0.681 | $\boxed{-25.414}$ |
| CDU in HH | 1.938 | 3 | 3.197 | −0.197 | −6.161 |
| CDU in NI | 12.597 | 21 | 20.785 | 0.215 | 1.033 |
| CDU in HB | 0.639 | 1 | 1.055 | −0.055 | −5.192 |
| CDU in BB | 2.662 | 4 | 4.392 | −0.392 | −8.932 |
| CDU in ST | 2.291 | 3 | 3.781 | $\boxed{-0.781}$ | −20.647 |
| CDU in BE | 3.180 | 5 | 5.248 | −0.248 | −4.722 |
| CDU in NW | 28.318 | 49 | 46.725 | $\boxed{2.275}$ | $\boxed{4.868}$ |
| CDU in SN | 4.530 | 7 | 7.475 | −0.475 | −6.358 |
| CDU in HE | 9.234 | 15 | 15.236 | −0.236 | −1.547 |
| CDU in TH | 2.198 | 3 | 3.626 | −0.626 | −17.270 |
| CDU in RP | 6.793 | 11 | 11.209 | −0.209 | −1.867 |
| CDU in BY | – | – | – | – | – |
| CDU in BW | 17.924 | 31 | 29.575 | 1.425 | 4.818 |
| CDU in SL | 1.439 | 2 | 2.374 | −0.374 | −15.765 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 165 | 165.000 | $\boxed{3.055}$ | $\boxed{30.282}$ |

| | | Using **Absolute** optimization | | | |
|---|---|---|---|---|---|
| | Proportion of votes (quota), in % | Seats (apportionment) | Number of seats according to the quota | Absolute deviation from the quota, in number of seats | Relative deviation from the quota, in % of the quota |
| | $\mathbf{q}*100$ | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})\,./\mathbf{q}*100$ |
| | 30 | 33 | 34 | 35 | 36 |
| CDU in SH | 4.630 | 8 | 7.640 | 0.360 | 4.712 |
| CDU in MV | 1.625 | 3 | 2.681 | 0.319 | $\boxed{11.879}$ |
| CDU in HH | 1.938 | 3 | 3.197 | −0.197 | −6.161 |
| CDU in NI | 12.597 | 21 | 20.785 | 0.215 | 1.033 |
| CDU in HB | 0.639 | 1 | 1.055 | −0.055 | −5.192 |
| CDU in BB | 2.662 | 4 | 4.392 | −0.392 | −8.932 |
| CDU in ST | 2.291 | 4 | 3.781 | 0.219 | 5.804 |
| CDU in BE | 3.180 | 5 | 5.248 | −0.248 | −4.722 |
| CDU in NW | 28.318 | 47 | 46.725 | 0.275 | 0.588 |
| CDU in SN | 4.530 | 7 | 7.475 | $\boxed{-0.475}$ | −6.358 |
| CDU in HE | 9.234 | 15 | 15.236 | −0.236 | −1.547 |
| CDU in TH | 2.198 | 4 | 3.626 | 0.374 | 10.307 |
| CDU in RP | 6.793 | 11 | 11.209 | −0.209 | −1.867 |
| CDU in BY | – | – | – | – | – |
| CDU in BW | 17.924 | 30 | 29.575 | $\boxed{0.425}$ | 1.437 |
| CDU in SL | 1.439 | 2 | 2.374 | −0.374 | $\boxed{-15.765}$ |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 165 | 165.000 | $\boxed{0.900}$ | $\boxed{27.644}$ |

| | | Using **Relative** optimization | | | |
|---|---|---|---|---|---|
| | Proportion of votes (quota), in % | Seats (apportionment) | Number of seats according to the quota | Absolute deviation from the quota, in number of seats | Relative deviation from the quota, in % of the quota |
| | $\mathbf{q}*100$ | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})\,./\mathbf{q}*100$ |
| | 40 | 43 | 44 | 45 | 46 |
| CDU in SH | 4.630 | 8 | 7.640 | 0.360 | 4.712 |
| CDU in MV | 1.625 | 3 | 2.681 | 0.319 | $\boxed{11.879}$ |
| CDU in HH | 1.938 | 3 | 3.197 | −0.197 | −6.161 |
| CDU in NI | 12.597 | 21 | 20.785 | 0.215 | 1.033 |
| CDU in HB | 0.639 | 1 | 1.055 | −0.055 | −5.192 |
| CDU in BB | 2.662 | 4 | 4.392 | −0.392 | −8.932 |
| CDU in ST | 2.291 | 4 | 3.781 | 0.219 | 5.804 |
| CDU in BE | 3.180 | 5 | 5.248 | −0.248 | −4.722 |
| CDU in NW | 28.318 | 47 | 46.725 | 0.275 | 0.588 |
| CDU in SN | 4.530 | 7 | 7.475 | $\boxed{-0.475}$ | −6.358 |
| CDU in HE | 9.234 | 15 | 15.236 | −0.236 | −1.547 |
| CDU in TH | 2.198 | 4 | 3.626 | 0.374 | 10.307 |
| CDU in RP | 6.793 | 11 | 11.209 | −0.209 | −1.867 |
| CDU in BY | – | – | – | – | – |
| CDU in BW | 17.924 | 30 | 29.575 | $\boxed{0.425}$ | 1.437 |
| CDU in SL | 1.439 | 2 | 2.374 | −0.374 | $\boxed{-15.765}$ |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 165 | 165.000 | $\boxed{0.900}$ | $\boxed{27.644}$ |

mandates between federal states. The table's layout and column numbers are the same as in Table 6 but the columns irrelevant to the current task are omitted.

Columns * 0 show the CDU's federal state quotas $q_j$, where $j$ is the federal state index. The federal state quotas are derived from the data in Table 7 and differ from one party to another. For example, the quota for the CDU's Schleswig-Holstein (SH) association is equal

$$q_{\mathrm{SH,\,CDU}} = \frac{518424}{11196374} \approx 0.04630 \quad (= 4.630\%)\ .$$

The frames in Table 9 highlight the apportionment accuracy. Here, the Sainte-Laguë method performs better than the D'Hondt method resulting in the accuracy of 0.558 against 2.275 seats, but both do not provide the desired accuracy of 0.5 seat achieved by the optimization methods. It should be however noted that, according to Tables 6 and 9, the Sainte-Laguë method distributes 164 seats, whereas other methods deal with 165 seats. Therefore, our comparison requires certain reservations.

We do not display similar tables for other Bundestag parties but show four distributions of Bundestag mandates between federal state party associations in Table 10. Here, the frames highlight the differences from the official Sainte-Laguë distribution in Columns 101–116, which is as in Rows 'Zusammen' (= Together) in the official Table 5. The output of Absolute optimization is very similar to that obtained using the Sainte-Laguë method: one GRÜNE's seat is redistributed from the federal state of Baden-Württemberg (BW) to Brandenburg (BB), and one CSU's Bavarian seat is redistributed to the CDU in Baden-Württemberg — but this is due to the difference between the total number of the CDU and CSU mandates; see Columns 'Total'.

Reducing the number of constituency election winners in Table 8 to the limits prescribed by Table 10, we obtain distributions of direct mandates in Table 11. Our distribution obtained using the Sainte-Laguë method confirms that in the official report; see Rows *'Wahlkreis'* (= Constituency) in Table 5. The optimal distributions of direct mandates are however a little bit different.

# 5 One-tier distribution of Bundestag mandates between federal state party associations

Thus, the official distribution of Bundestag mandates among federal state party associations is two-tier: first, the Bundestag seats are distributed between the eligible parties in proportion to the votes cast for the parties nationwide, and next the mandates of each party are distributed between federal states. At each step, the parties' fractional quotas are converted into integer number of mandates, which among other things introduces rounding errors. To avoid their two-tier cumulation, the Bundestag mandates can be distributed in one step in proportion to the votes cast for the parties in federal states as given in Table 7. The computation models are exactly the same as in Sections 3 and 4 with the only difference that instead of seven parties or 16 federal states we consider a total of 81 federal state party associations: of the SPD, AfD, GRÜNE and LINKE in all the 16 states, of the CDU in all the states except for Bavaria, of the CSU in Bavaria, and of the SSW in Schleswig-Holstein.

For comparisons, the two-tier and one-tier distributions of mandates are displayed in Table 12. Column 1, 'Proportion of votes (sub-quotas), in %', shows the percentage of votes cast for the party in the federal state reduced to the total of votes for the seven Bundestag parties; therefore, the sum of these percentages is equal to 100 at the bottom of the table at its second page. Section 'Two-tier distribution of seats' (Columns 2–5) is a reshaped version of Table 10. In particular, Column 2 shows the distribution of Bundestag seats using the official two-tier Sainte-Laguë method. The output of other models is shown in Columns 3–5, where the differences from the official Sainte-Laguë distribution are highlighted by fames.

The new Section 'One-tier distribution of seats' includes Columns 6–9. As follows from the summary at the bottom of the table (in the second page), the one-tier distributions are more accurate. For example, the official two-tier Sainte-Laguë distribution in Column 2 has the accuracy within interval $[-0.733,\ 0.520]$

Table 10: Distribution of Bundestag mandates between federal state party associations. The differences from the official Sainte-Laguë distribution are highlighted by frames.

| | | Using **Sainte-Laguë** method | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Total | SH | MV | HH | NI | HB | BB | ST | BE | NW | SN | HE | TH | RP | BY | BW | SL |
| | | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 | 111 | 112 | 113 | 114 | 115 | 116 |
| CDU | 164 | 8 | 3 | 3 | 21 | 1 | 4 | 4 | 5 | 47 | 7 | 15 | 4 | 11 | – | 29 | 2 |
| AfD | 152 | 5 | 5 | 2 | 13 | 1 | 8 | 7 | 4 | 26 | 14 | 9 | 8 | 7 | 22 | 19 | 2 |
| SPD | 120 | 5 | 2 | 3 | 17 | 1 | 4 | 2 | 4 | 31 | 3 | 10 | 2 | 7 | 14 | 13 | 2 |
| GRÜNE | 85 | 4 | 1 | 3 | 8 | 1 | 2 | 1 | 5 | 19 | 2 | 7 | 1 | 4 | 14 | 12 | 1 |
| LINKE | 64 | 2 | 2 | 2 | 6 | 1 | 3 | 2 | 6 | 13 | 4 | 4 | 3 | 2 | 7 | 6 | 1 |
| CSU | 44 | – | – | – | – | – | – | – | – | – | – | – | – | – | 44 | – | – |
| SSW | 1 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

| | | Using **D'Hondt** method | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Total | SH | MV | HH | NI | HB | BB | ST | BE | NW | SN | HE | TH | RP | BY | BW | SL |
| | | 201 | 202 | 203 | 204 | 205 | 206 | 207 | 208 | 209 | 210 | 211 | 212 | 213 | 214 | 215 | 216 |
| CDU | $\boxed{165}$ | 8 | $\boxed{2}$ | 3 | 21 | 1 | 4 | $\boxed{3}$ | 5 | $\boxed{49}$ | 7 | 15 | $\boxed{3}$ | 11 | – | $\boxed{31}$ | 2 |
| AfD | 152 | $\boxed{4}$ | 5 | $\boxed{1}$ | $\boxed{14}$ | $\boxed{0}$ | 8 | 7 | 4 | $\boxed{27}$ | $\boxed{15}$ | 9 | $\boxed{7}$ | 7 | $\boxed{23}$ | 19 | 2 |
| SPD | 120 | 5 | $\boxed{1}$ | 3 | $\boxed{18}$ | 1 | $\boxed{3}$ | 2 | 4 | $\boxed{32}$ | 3 | 10 | $\boxed{1}$ | 7 | 14 | $\boxed{14}$ | 2 |
| GRÜNE | 85 | 4 | $\boxed{0}$ | 3 | $\boxed{9}$ | $\boxed{0}$ | $\boxed{1}$ | $\boxed{0}$ | 5 | $\boxed{21}$ | 2 | 7 | $\boxed{0}$ | 4 | $\boxed{15}$ | $\boxed{14}$ | $\boxed{0}$ |
| LINKE | 64 | 2 | 2 | 2 | 6 | $\boxed{0}$ | $\boxed{2}$ | 2 | 6 | $\boxed{14}$ | 4 | $\boxed{5}$ | 3 | 2 | 7 | $\boxed{7}$ | $\boxed{0}$ |
| CSU | $\boxed{43}$ | – | – | – | – | – | – | – | – | – | – | – | – | – | $\boxed{43}$ | – | – |
| SSW | 1 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

| | | Using **Absolute** optimization | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Total | SH | MV | HH | NI | HB | BB | ST | BE | NW | SN | HE | TH | RP | BY | BW | SL |
| | | 301 | 302 | 303 | 304 | 305 | 306 | 307 | 308 | 309 | 310 | 311 | 312 | 313 | 314 | 315 | 316 |
| CDU | $\boxed{165}$ | 8 | 3 | 3 | 21 | 1 | 4 | 4 | 5 | 47 | 7 | 15 | 4 | 11 | – | $\boxed{30}$ | 2 |
| AfD | 152 | 5 | 5 | 2 | 13 | 1 | 8 | 7 | 4 | 26 | 14 | 9 | 8 | 7 | 22 | 19 | 2 |
| SPD | 120 | 5 | 2 | 3 | 17 | 1 | 4 | 2 | 4 | 31 | 3 | 10 | 2 | 7 | 14 | 13 | 2 |
| GRÜNE | 85 | 4 | 1 | 3 | 8 | 1 | $\boxed{1}$ | 1 | 5 | 19 | 2 | 7 | 1 | 4 | 14 | $\boxed{13}$ | 1 |
| LINKE | 64 | 2 | 2 | 2 | 6 | 1 | 3 | 2 | 6 | 13 | 4 | 4 | 3 | 2 | 7 | 6 | 1 |
| CSU | $\boxed{43}$ | – | – | – | – | – | – | – | – | – | – | – | – | – | $\boxed{43}$ | – | – |
| SSW | 1 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

| | | Using **Relative** optimization | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Total | SH | MV | HH | NI | HB | BB | ST | BE | NW | SN | HE | TH | RP | BY | BW | SL |
| | | 401 | 402 | 403 | 404 | 405 | 406 | 407 | 408 | 409 | 410 | 411 | 412 | 413 | 414 | 415 | 416 |
| CDU | $\boxed{165}$ | 8 | 3 | 3 | 21 | 1 | 4 | 4 | 5 | 47 | 7 | 15 | 4 | 11 | – | $\boxed{30}$ | 2 |
| AfD | 152 | 5 | $\boxed{6}$ | 2 | 13 | 1 | 8 | 7 | $\boxed{5}$ | $\boxed{25}$ | 14 | 9 | 8 | 7 | 22 | $\boxed{18}$ | 2 |
| SPD | 120 | 5 | 2 | 3 | 17 | 1 | 4 | 2 | 4 | 31 | 3 | 10 | 2 | 7 | 14 | 13 | 2 |
| GRÜNE | 85 | 4 | 1 | 3 | 8 | 1 | 2 | 1 | 5 | $\boxed{18}$ | $\boxed{3}$ | 7 | 1 | 4 | 14 | 12 | 1 |
| LINKE | 64 | 2 | 2 | 2 | 6 | 1 | 3 | 2 | 6 | $\boxed{12}$ | 4 | $\boxed{5}$ | 3 | $\boxed{3}$ | $\boxed{6}$ | 6 | 1 |
| CSU | $\boxed{43}$ | – | – | – | – | – | – | – | – | – | – | – | – | – | $\boxed{43}$ | – | – |
| SSW | 1 | 1 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

Table 11: Number of hypothetical direct mandates by federal state and party. The differences from that actually issued are indicated by frames. The federal state name abbreviations are explained in Table 7.

| | | Using **Sainte-Laguë** method | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Total | SH | MV | HH | NI | HB | BB | ST | BE | NW | SN | HE | TH | RP | BY | BW | SL |
| | | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 | 111 | 112 | 113 | 114 | 115 | 116 |
| CDU | 128 | 8 | – | 1 | 15 | – | – | – | 3 | 44 | – | 15 | – | 11 | – | 29 | 2 |
| AfD | 42 | – | 5 | – | – | – | 8 | 7 | 1 | – | 14 | – | 7 | – | – | – | – |
| SPD | 44 | 1 | – | 3 | 15 | 1 | 1 | – | 1 | 17 | – | 2 | – | 1 | – | – | 2 |
| GRÜNE | 12 | 1 | – | 2 | – | – | – | – | 3 | 3 | – | – | – | – | – | 3 | – |
| LINKE | 6 | – | – | – | – | – | – | – | 4 | – | 1 | – | 1 | – | – | – | – |
| CSU | 44 | – | – | – | – | – | – | – | – | – | – | – | – | – | 44 | – | – |
| SSW | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

| | | Using **D'Hondt** method | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Total | SH | MV | HH | NI | HB | BB | ST | BE | NW | SN | HE | TH | RP | BY | BW | SL |
| | | 201 | 202 | 203 | 204 | 205 | 206 | 207 | 208 | 209 | 210 | 211 | 212 | 213 | 214 | 215 | 216 |
| CDU | $\boxed{130}$ | 8 | – | 1 | 15 | – | – | – | 3 | 44 | – | 15 | – | 11 | – | $\boxed{31}$ | 2 |
| AfD | $\boxed{43}$ | – | 5 | – | – | – | 8 | 7 | 1 | – | $\boxed{15}$ | – | 7 | – | – | – | – |
| SPD | 44 | 1 | – | 3 | 15 | 1 | 1 | – | 1 | 17 | – | 2 | – | 1 | – | – | 2 |
| GRÜNE | 12 | 1 | – | 2 | – | – | – | – | 3 | 3 | – | – | – | – | – | 3 | – |
| LINKE | 6 | – | – | – | – | – | – | – | 4 | – | 1 | – | 1 | – | – | – | – |
| CSU | $\boxed{43}$ | – | – | – | – | – | – | – | – | – | – | – | – | – | $\boxed{43}$ | – | – |
| SSW | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

| | | Using **Absolute** optimization | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Total | SH | MV | HH | NI | HB | BB | ST | BE | NW | SN | HE | TH | RP | BY | BW | SL |
| | | 301 | 302 | 303 | 304 | 305 | 306 | 307 | 308 | 309 | 310 | 311 | 312 | 313 | 314 | 315 | 316 |
| CDU | $\boxed{129}$ | 8 | – | 1 | 15 | – | – | – | 3 | 44 | – | 15 | – | 11 | – | $\boxed{30}$ | 2 |
| AfD | 42 | – | 5 | – | – | – | 8 | 7 | 1 | – | 14 | – | 7 | – | – | – | – |
| SPD | 44 | 1 | – | 3 | 15 | 1 | 1 | – | 1 | 17 | – | 2 | – | 1 | – | – | 2 |
| GRÜNE | 12 | 1 | – | 2 | – | – | – | – | 3 | 3 | – | – | – | – | – | 3 | – |
| LINKE | 6 | – | – | – | – | – | – | – | 4 | – | 1 | – | 1 | – | – | – | – |
| CSU | $\boxed{43}$ | – | – | – | – | – | – | – | – | – | – | – | – | – | $\boxed{43}$ | – | – |
| SSW | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

| | | Using **Relative** optimization | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Total | SH | MV | HH | NI | HB | BB | ST | BE | NW | SN | HE | TH | RP | BY | BW | SL |
| | | 401 | 402 | 403 | 404 | 405 | 406 | 407 | 408 | 409 | 410 | 411 | 412 | 413 | 414 | 415 | 416 |
| CDU | $\boxed{129}$ | 8 | – | 1 | 15 | – | – | – | 3 | 44 | – | 15 | – | 11 | – | $\boxed{30}$ | 2 |
| AfD | $\boxed{43}$ | – | $\boxed{6}$ | – | – | – | 8 | 7 | 1 | – | 14 | – | 7 | – | – | – | – |
| SPD | 44 | 1 | – | 3 | 15 | 1 | 1 | – | 1 | 17 | – | 2 | – | 1 | – | – | 2 |
| GRÜNE | 12 | 1 | – | 2 | – | – | – | – | 3 | 3 | – | – | – | – | – | 3 | – |
| LINKE | 6 | – | – | – | – | – | – | – | 4 | – | 1 | – | 1 | – | – | – | – |
| CSU | $\boxed{43}$ | – | – | – | – | – | – | – | – | – | – | – | – | – | $\boxed{43}$ | – | – |
| SSW | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

Table 12: Two- and one-tier distributions of Bundestag mandates between federal state party associations

| | Proportion of votes (sub-quotas), in % | Two-tier distribution of seats | | | | One-tier distribution of seats | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | Using **Sainte-Laguë** method | Using **D'Hondt** method | Using **Absolute** optimization | Using **Relative** optimization | Using **Sainte-Laguë** method | Using **D'Hondt** method | Using **Absolute** optimization | Using **Relative** optimization |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| CDU in SH | 1.210 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| CDU in MV | 0.425 | 3 | $\boxed{2}$ | 3 | 3 | 3 | $\boxed{2}$ | 3 | 3 |
| CDU in HH | 0.506 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| CDU in NI | 3.293 | 21 | 21 | 21 | 21 | 21 | $\boxed{22}$ | 21 | 21 |
| CDU in HB | 0.167 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| CDU in BB | 0.696 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| CDU in ST | 0.599 | 4 | $\boxed{3}$ | 4 | 4 | 4 | 4 | 4 | 4 |
| CDU in BE | 0.831 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| CDU in NW | 7.402 | 47 | $\boxed{49}$ | 47 | 47 | 47 | $\boxed{49}$ | 47 | 47 |
| CDU in SN | 1.184 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| CDU in HE | 2.414 | 15 | 15 | 15 | 15 | 15 | $\boxed{16}$ | 15 | 15 |
| CDU in TH | 0.574 | 4 | $\boxed{3}$ | 4 | 4 | 4 | $\boxed{3}$ | 4 | 4 |
| CDU in RP | 1.776 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 |
| CDU in BW | 4.685 | 29 | $\boxed{31}$ | $\boxed{30}$ | $\boxed{30}$ | 29 | $\boxed{31}$ | $\boxed{30}$ | 29 |
| CDU in SL | 0.376 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| AfD in SH | 0.715 | 5 | $\boxed{4}$ | 5 | 5 | $\boxed{4}$ | $\boxed{4}$ | $\boxed{4}$ | $\boxed{4}$ |
| AfD in MV | 0.834 | 5 | 5 | 5 | $\boxed{6}$ | 5 | 5 | 5 | 5 |
| AfD in HH | 0.265 | 2 | $\boxed{1}$ | 2 | 2 | 2 | $\boxed{1}$ | 2 | 2 |
| AfD in NI | 2.088 | 13 | $\boxed{14}$ | 13 | 13 | 13 | $\boxed{14}$ | 13 | 13 |
| AfD in HB | 0.123 | 1 | $\boxed{-}$ | 1 | 1 | 1 | $\boxed{-}$ | 1 | 1 |
| AfD in BB | 1.250 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| AfD in ST | 1.158 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| AfD in BE | 0.693 | 4 | 4 | 4 | $\boxed{5}$ | 4 | 4 | 4 | 4 |
| AfD in NW | 4.133 | 26 | $\boxed{27}$ | 26 | $\boxed{25}$ | 26 | $\boxed{27}$ | 26 | 26 |
| AfD in SN | 2.238 | 14 | $\boxed{15}$ | 14 | 14 | 14 | $\boxed{15}$ | 14 | 14 |
| AfD in HE | 1.487 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| AfD in TH | 1.192 | 8 | $\boxed{7}$ | 8 | 8 | 8 | $\boxed{7}$ | $\boxed{7}$ | $\boxed{7}$ |
| AfD in RP | 1.164 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| AfD in BY | 3.539 | 22 | $\boxed{23}$ | 22 | 22 | 22 | $\boxed{23}$ | 22 | 22 |
| AfD in BW | 2.933 | 19 | 19 | 19 | $\boxed{18}$ | $\boxed{18}$ | 19 | $\boxed{18}$ | $\boxed{18}$ |
| AfD in SL | 0.302 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| SPD in SH | 0.823 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| SPD in MV | 0.296 | 2 | $\boxed{1}$ | 2 | 2 | 2 | $\boxed{1}$ | 2 | 2 |
| SPD in HH | 0.555 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| SPD in NI | 2.693 | 17 | $\boxed{18}$ | 17 | 17 | 17 | $\boxed{18}$ | 17 | 17 |
| SPD in HB | 0.188 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| SPD in BB | 0.570 | 4 | $\boxed{3}$ | 4 | 4 | 4 | $\boxed{3}$ | 4 | 4 |
| SPD in ST | 0.342 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| SPD in BE | 0.689 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| SPD in NW | 4.922 | 31 | $\boxed{32}$ | 31 | 31 | 31 | $\boxed{33}$ | 31 | 31 |
| SPD in SN | 0.507 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| SPD in HE | 1.535 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| SPD in TH | 0.271 | 2 | $\boxed{1}$ | 2 | 2 | 2 | $\boxed{1}$ | 2 | 2 |
| SPD in RP | 1.080 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| SPD in BY | 2.149 | 14 | 14 | 14 | 14 | 14 | 14 | 14 | $\boxed{13}$ |
| SPD in BW | 2.098 | 13 | $\boxed{14}$ | 13 | 13 | 13 | $\boxed{14}$ | 13 | 13 |
| SPD in SL | 0.306 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

*(Continued next page…)*

Table 12: **(continued)** Two- and one-tier distributions of Bundestag mandates between federal state party associations

| | Proportion of votes (sub-quotas), in % | Two-tier distribution of seats | | | | One-tier distribution of seats | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | Using **Sainte-Laguë** method | Using **D'Hondt** method | Using **Absolute** optimization | Using **Relative** optimization | Using **Sainte-Laguë** method | Using **D'Hondt** method | Using **Absolute** optimization | Using **Relative** optimization |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| GRÜNE in SH | 0.654 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| GRÜNE in MV | 0.128 | 1 | $\boxed{–}$ | 1 | 1 | 1 | $\boxed{–}$ | 1 | 1 |
| GRÜNE in HH | 0.471 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| GRÜNE in NI | 1.347 | 8 | $\boxed{9}$ | 8 | 8 | 8 | $\boxed{9}$ | 8 | 8 |
| GRÜNE in HB | 0.127 | 1 | $\boxed{–}$ | 1 | 1 | 1 | $\boxed{–}$ | 1 | 1 |
| GRÜNE in BB | 0.254 | 2 | $\boxed{1}$ | $\boxed{1}$ | 2 | 2 | $\boxed{1}$ | 2 | 2 |
| GRÜNE in ST | 0.138 | 1 | $\boxed{–}$ | 1 | 1 | 1 | $\boxed{–}$ | 1 | 1 |
| GRÜNE in BE | 0.766 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| GRÜNE in NW | 3.037 | 19 | $\boxed{21}$ | 19 | $\boxed{18}$ | 19 | $\boxed{20}$ | 19 | 19 |
| GRÜNE in SN | 0.391 | 2 | 2 | 2 | $\boxed{3}$ | 2 | 2 | 2 | $\boxed{3}$ |
| GRÜNE in HE | 1.054 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| GRÜNE in TH | 0.131 | 1 | $\boxed{–}$ | 1 | 1 | 1 | $\boxed{–}$ | 1 | 1 |
| GRÜNE in RP | 0.600 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| GRÜNE in BY | 2.235 | 14 | $\boxed{15}$ | 14 | 14 | 14 | 14 | 14 | 14 |
| GRÜNE in BW | 2.021 | 12 | $\boxed{14}$ | $\boxed{13}$ | 12 | $\boxed{13}$ | $\boxed{13}$ | $\boxed{13}$ | $\boxed{13}$ |
| GRÜNE in SL | 0.101 | 1 | $\boxed{–}$ | 1 | 1 | 1 | $\boxed{–}$ | 1 | 1 |
| LINKE in SH | 0.342 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| LINKE in MV | 0.287 | 2 | 2 | 2 | 2 | 2 | $\boxed{1}$ | 2 | 2 |
| LINKE in HH | 0.353 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| LINKE in NI | 0.947 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| LINKE in HB | 0.120 | 1 | $\boxed{–}$ | 1 | 1 | 1 | $\boxed{–}$ | 1 | 1 |
| LINKE in BB | 0.411 | 3 | $\boxed{2}$ | 3 | 3 | 3 | $\boxed{2}$ | 3 | 3 |
| LINKE in ST | 0.336 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| LINKE in BE | 0.904 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| LINKE in NW | 2.048 | 13 | $\boxed{14}$ | 13 | $\boxed{12}$ | 13 | 13 | 13 | 13 |
| LINKE in SN | 0.678 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| LINKE in HE | 0.726 | 4 | $\boxed{5}$ | 4 | $\boxed{5}$ | $\boxed{5}$ | 4 | $\boxed{5}$ | $\boxed{5}$ |
| LINKE in TH | 0.469 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| LINKE in RP | 0.378 | 2 | 2 | 2 | $\boxed{3}$ | 2 | 2 | 2 | $\boxed{3}$ |
| LINKE in BY | 1.067 | 7 | 7 | 7 | $\boxed{6}$ | 7 | 7 | 7 | 7 |
| LINKE in BW | 1.003 | 6 | $\boxed{7}$ | 6 | 6 | 6 | 6 | 6 | 6 |
| LINKE in SL | 0.103 | 1 | $\boxed{–}$ | 1 | 1 | 1 | $\boxed{–}$ | 1 | 1 |
| CSU in BY | 6.920 | 44 | $\boxed{43}$ | $\boxed{43}$ | $\boxed{43}$ | 44 | $\boxed{46}$ | 44 | 44 |
| SSW in SH | 0.178 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Total | 100.000 | 630 | 630 | 630 | 630 | 630 | 630 | 630 | 630 |
| Maximum negative/ positive absolute deviations from sub-quotas, in seats | | −0.733<br>0.520 | −0.869<br>2.366 | −0.597<br>0.520 | −1.134<br>0.744 | −0.517<br>0.491 | −0.869<br>2.405 | −0.509<br>0.483 | −0.541<br>0.619 |
| Maximum negative/ positive relative deviations from sub-quotas, in % | | −18.706<br>56.762 | −100.000<br>10.813 | −37.393<br>56.762 | −15.650<br>56.762 | −18.706<br>56.762 | −100.000<br>6.413 | −18.706<br>56.762 | −15.650<br>56.762 |
| Range of adjustment vote weights | | 0.928 | ∞ | 1.504 | 0.858 | 0.928 | ∞ | 0.928 | 0.858 |

seats, whereas the one-tier Saite-Laguë distribution in Column 6 demonstrates the accuracy within interval $[-0.517,\ 0.491]$. The one-tier Absolute optimization in Column 8 results in the desired accuracy of $\pm 0.5$ seat, which is not achievable using two-tier methods.

It should be noted that the D'Hondt method allocates no Bundestag seat to the eight federal state party associations with sub-quotas less than 0.138%.[6] At the same time, the CDU in Nordrhein-Westfalia (NW) and Baden-Württemberg (BW) with sub-quotas of 7.402% and 4.685%, respectively, get two seats more than in the official distribution. We observe the known property of the D'Hondt method to favor the strong at the expense of the weak; this is why it was replaced by the less biased Sainte-Laguë method.

Let us see how the one-tier distributions of Bundestag seats between federal state party associations meet the nationwide quotas derived from the nationwide votes. For this purpose, we summarize the mandates of each party in Columns 6–9 of Table 12 and analyze the resulting Bundestag apportionments in Table 13. These apportionments differ from that in Table 6, because they are not fit to the nationwide quotas but to the federal state sub-quotas, i.e. calculated with another model. Here, the section titles like 'Using Sainte-Laguë method' or 'Using Relative optimization' do not mean that the respective apportionment is calculated using the method indicated. This only applies to the one-tier mandate distribution between federal state party associations from which it originates. In particular, this explains why the apportionment in Section 'Relative optimization' can be not optimal from the viewpoint of relative optimization — indeed, the relative optimization criterion is better fulfilled by the apportionment from Section 'Absolute optimization'; cf. Columns 36 and 46.

We see that the Bundestag apportionment can be based either on nationwide or federal state votes. The officially accepted first option leans toward the agent concept of representation, while the second option emphasizes the descriptive concept. It is also possible to define parties' federal state sub-quotas not only in proportion to votes they receive in federal states but with taking into account the size of their electorate, or the number of their citizens including minors, or even the total of their inhabitants; then the voters act as their collective representatives. The reason is that taking into account only votes causes underrepresentation of federal states with a low turnout in elections — a further deviation from the descriptive concept of representation.

# 6 Adjustment vote weights for making the Bundestag more inclusive

The 2023/24 electoral reform, having restricted the Bundestag size, left 23 out of 299 constituencies with no representative in the 2025 Bundestag. Thereby, the reform made it less inclusive for local representatives. At the conceptual level, this means a significant bias toward the agent approach to political representation at the expense of the descriptive one.

Before the 2023/24 reform, all constituency election winners received Bundestag mandates. Let us see how large the 2025 Bundestag would be if all constituency election winners received direct mandates. In this case, for each party $i$, its quota $q_i$ expressed in the rounded number of seats in the Bundestag of size $S$ must exceed the number of the party's direct mandates $d_i$:

$$q_i S + 0.5 \geq d_i \quad \Leftrightarrow \quad S \geq \frac{d_i - 0.5}{q_i}\ , \quad i = \text{CDU},\ \ldots,\ \text{SSW}\ .$$

Taking the data from Column 'Total' of Table 8 and Column 10 of Table 6, we represent these inequalities for seven parties in the vector form:

$$S \quad \geq \quad \max[(\mathbf{d} - 0.5)./\mathbf{q}]$$

[6]For the federal state party associations without Bundestag mandates, the relative deviation from the sub-quota is obviously $-100\%$, and the adjustment vote weight is equal to $\infty$.

Table 13: Total party mandates in their one-tier distributions between federal states

Using **Sainte-Laguë** method

| | Proportion of votes (quota), in % | Seats (apportionment) | Number of seats according to the quota | Absolute deviation from the quota, in number of seats | Relative deviation from the quota, in % of the quota | Adjustment vote weight | Party's votes (Seats × Vote weight) | Share of power (Seats × Vote weight), in % |
|---|---|---|---|---|---|---|---|---|
| | $\mathbf{q}*100$ | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'*\mathbf{w}}*100$ |
| | 10 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| CDU | 26.139 | 164 | 164.678 | −0.678 | −0.412 | 1.019 | $\boxed{167.099}$ | → 26.139 |
| AfD | 24.114 | 150 | 151.917 | $\boxed{-1.917}$ | −1.262 | 1.028 | 154.151 | → 24.114 |
| SPD | 19.025 | 120 | 119.859 | 0.141 | 0.118 | 1.014 | 121.621 | → 19.025 |
| GRÜNE | 13.453 | 86 | 84.754 | $\boxed{1.246}$ | $\boxed{1.470}$ | $\boxed{1.000}$ | 86.000 | → 13.453 |
| LINKE | 10.171 | 65 | 64.077 | 0.923 | 1.441 | 1.000 | 65.019 | → 10.171 |
| CSU | 6.920 | 44 | 43.595 | 0.405 | 0.928 | 1.005 | 44.236 | → 6.920 |
| SSW | 0.178 | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.136}$ | $\boxed{1.136}$ | → 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 630 | 630.000 | $\boxed{3.163}$ | $\boxed{12.172}$ | $\boxed{0.136}$ | $\boxed{165.963}$ | → 100.000 |

Using **D'Hondt** method

| | $\mathbf{q}*100$ | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'*\mathbf{w}}*100$ |
|---|---|---|---|---|---|---|---|---|
| | 20 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| CDU | 26.139 | 168 | 164.678 | $\boxed{3.322}$ | 2.017 | 1.034 | $\boxed{173.761}$ | → 26.139 |
| AfD | 24.114 | 152 | 151.917 | 0.083 | 0.054 | 1.055 | 160.297 | → 24.114 |
| SPD | 19.025 | 121 | 119.859 | 1.141 | 0.952 | 1.045 | 126.470 | → 19.025 |
| GRÜNE | 13.453 | 82 | 84.754 | −2.754 | −3.249 | 1.091 | 89.429 | → 13.453 |
| LINKE | 10.171 | 60 | 64.077 | $\boxed{-4.077}$ | −6.362 | 1.127 | 67.611 | → 10.171 |
| CSU | 6.920 | 46 | 43.595 | 2.405 | $\boxed{5.516}$ | $\boxed{1.000}$ | 46.000 | → 6.920 |
| SSW | 0.178 | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.182}$ | $\boxed{1.182}$ | → 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 630 | 630.000 | $\boxed{7.398}$ | $\boxed{16.218}$ | $\boxed{0.182}$ | $\boxed{172.580}$ | → 100.000 |

Using **Absolute** optimization

| | $\mathbf{q}*100$ | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'*\mathbf{w}}*100$ |
|---|---|---|---|---|---|---|---|---|
| | 30 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| CDU | 26.139 | 165 | 164.678 | 0.322 | 0.195 | 1.013 | $\boxed{167.099}$ | → 26.139 |
| AfD | 24.114 | 149 | 151.917 | $\boxed{-2.917}$ | −1.920 | 1.035 | 154.151 | → 24.114 |
| SPD | 19.025 | 120 | 119.859 | 0.141 | 0.118 | 1.014 | 121.621 | → 19.025 |
| GRÜNE | 13.453 | 86 | 84.754 | $\boxed{1.246}$ | $\boxed{1.470}$ | $\boxed{1.000}$ | 86.000 | → 13.453 |
| LINKE | 10.171 | 65 | 64.077 | 0.923 | 1.441 | 1.000 | 65.019 | → 10.171 |
| CSU | 6.920 | 44 | 43.595 | 0.405 | 0.928 | 1.005 | 44.236 | → 6.920 |
| SSW | 0.178 | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.136}$ | $\boxed{1.136}$ | → 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 630 | 630.000 | $\boxed{4.163}$ | $\boxed{12.172}$ | $\boxed{0.136}$ | $\boxed{165.963}$ | → 100.000 |

Using **Relative** optimization

| | $\mathbf{q}*100$ | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'*\mathbf{w}}*100$ |
|---|---|---|---|---|---|---|---|---|
| | 40 | 43 | 44 | 45 | 46 | 47 | 48 | 49 |
| CDU | 26.139 | 164 | 164.678 | −0.678 | −0.412 | 1.034 | $\boxed{169.621}$ | → 26.139 |
| AfD | 24.114 | 149 | 151.917 | $\boxed{-2.917}$ | −1.920 | 1.050 | 156.478 | → 24.114 |
| SPD | 19.025 | 119 | 119.859 | −0.859 | −0.716 | 1.037 | 123.456 | → 19.025 |
| GRÜNE | 13.453 | 87 | 84.754 | $\boxed{2.246}$ | 2.650 | 1.003 | 87.298 | → 13.453 |
| LINKE | 10.171 | 66 | 64.077 | 1.923 | $\boxed{3.002}$ | $\boxed{1.000}$ | 66.000 | → 10.171 |
| CSU | 6.920 | 44 | 43.595 | 0.405 | 0.928 | 1.021 | 44.904 | → 6.920 |
| SSW | 0.178 | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.153}$ | $\boxed{1.153}$ | → 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 630 | 630.000 | $\boxed{5.163}$ | $\boxed{13.704}$ | $\boxed{0.153}$ | $\boxed{168.468}$ | → 100.000 |

$$
\begin{aligned}
&= \max\left[\begin{pmatrix} 143-0.5 \\ 46-0.5 \\ 45-0.5 \\ 12-0.5 \\ 6-0.5 \\ 47-0.5 \\ 0-0.5 \end{pmatrix} ./ \begin{pmatrix} 0.26139 \\ 0.24114 \\ 0.19025 \\ 0.13453 \\ 0.10171 \\ 0.06920 \\ 0.00178 \end{pmatrix}\right] \\
&= \max \underbrace{\left[\begin{pmatrix} 545.1624 \\ 188.6871 \\ 233.9028 \\ 85.4828 \\ 54.0753 \\ 671.9653 \\ -280.8989 \end{pmatrix}\right]}_{\min S \text{ for each party}} \\
&\approx 672\,. \qquad (1)
\end{aligned}
$$

If, contrary to this estimate, the Bundestag with 299 direct mandates is restricted to 630 seats — as prescribed by the 2023/24 reform — then the deviations from party quotas significantly exceed the rounding error of 0.5 seat. As follows from Table 14, in this case four apportionment methods equally allocate seats to parties with absolute deviations from the quotas in interval $[-0.917; 3.405]$ (Columns * 6 ). The CSU with its 47 direct mandates is the most overrepresented party; cf. with Table 6 where CSU gets 43 or 44 Bundestag seats. The side effect of the CSU's overrepresentation is the underrepresentation of other parties, especially of CDU, AfD, SPD and GRÜNE, whose negative deviations from their quotas exceed the tolerated 0.5 seat.

Thus, the descriptive concept of representation can be emphasized either at the expense of reducing in mandates of certain parties, that is, at the expense of the agent concept, or by increasing the Bundestag size. However, this alternative can be surmounted by relaxing the 'ideal one man—one vote'. In fact, the party power is determined by the number of its votes in the Bundestag, which is not necessarily linked to the number of mandates. As mentioned in Introduction, the vote power of each shareholder in a joint stock company is proportional to his/her percentage of shares. Similarly, members of parliament factions can have individual adjustment vote weights.

The idea of adjustment vote weight $w_i$ for a party $i$ with $x_i > 0$ seats (not mandates!) is to bring its voting power $w_i x_i$ into correspondence with the party quota $q_i$ in the Bundestag with $S$ seats:

$$
w_i x_i = q_i S \quad \Leftrightarrow \quad w_i = S q_i / x_i, \quad i = \text{CDU}, \ \dots\ , \ \text{SSW}\,.
$$

This way the adjustment vote weights are computed for all Bundestag factions. To simplify comparisons, we normalize the parties' adjustment vote weights by making the minimum weight equal to 1, that is, divide all weights by their minimum (then it is easy to see by which percent this or that vote weight deviates from the reference of one vote).

The adjustment vote weights in Table 14 for the parties in the Bundestag with 299 direct mandates are computed by the following vector formula:[7]

$$
\mathbf{w} \;=\; \frac{S\mathbf{q}./\mathbf{x}}{\min[S\mathbf{q}./\mathbf{x}]}
$$

[7]The adjustment vote weights below are computed for the quotas rounded to the third decimal. In Table 14, they are computed for non-rounded quotas, explaining why the SSW weight of $1.209 \neq 1.207$ in Table 14.

Table 14: Hypothetical inclusive Bundestag with 299 direct mandates

| | Proportion of votes (quota), in % | Direct mandates | | Adjustment seats | | Seats (apportionment) | Number of seats according to the quota | Absolute deviation from the quota, in number of seats | Relative deviation from the quota, in % of the quota | Adjustment vote weight | Party's votes (Seats × Vote weight) | | Share of power (Seats × Vote weight), in % |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Using **Sainte-Laguë** method | | | | | | | | | | | | |
| | $\mathbf{q}*100$ | $\mathbf{d}$ | | $\mathbf{a}$ | | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'*\mathbf{w}}*100$ |
| | 10 | 11 | | 12 | | 13 | 14 | 15 | 16 | 17 | 18 | | 19 |
| CDU | 26.139 | 143 | + | 21 | = | 164 | 164.678 | −0.678 | −0.412 | 1.083 | $\boxed{177.539}$ | → | 26.139 |
| AfD | 24.114 | 46 | + | 105 | = | 151 | 151.917 | $\boxed{-0.917}$ | −0.604 | 1.085 | 163.781 | → | 24.114 |
| SPD | 19.025 | 45 | + | 74 | = | 119 | 119.859 | −0.859 | −0.716 | 1.086 | 129.219 | → | 19.025 |
| GRÜNE | 13.453 | 12 | + | 72 | = | 84 | 84.754 | −0.754 | −0.890 | 1.088 | 91.373 | → | 13.453 |
| LINKE | 10.171 | 6 | + | 58 | = | 64 | 64.077 | −0.077 | −0.120 | 1.079 | 69.081 | → | 10.171 |
| CSU | 6.920 | 47 | | – | = | 47 | 43.595 | $\boxed{3.405}$ | $\boxed{7.810}$ | $\boxed{1.000}$ | 47.000 | → | 6.920 |
| SSW | 0.178 | – | | 1 | = | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.207}$ | $\boxed{1.207}$ | → | 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 299 | + | 331 | = | 630 | 630.000 | $\boxed{4.322}$ | $\boxed{18.512}$ | $\boxed{0.207}$ | $\boxed{176.331}$ | → | 100.000 |
| | Using **D'Hondt** method | | | | | | | | | | | | |
| | $\mathbf{q}*100$ | $\mathbf{d}$ | | $\mathbf{a}$ | | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}*100$ |
| | 20 | 21 | | 22 | | 23 | 24 | 25 | 26 | 27 | 28 | | 29 |
| CDU | 26.139 | 143 | + | 21 | = | 164 | 164.678 | −0.678 | −0.412 | 1.083 | $\boxed{177.539}$ | → | 26.139 |
| AfD | 24.114 | 46 | + | 105 | = | 151 | 151.917 | $\boxed{-0.917}$ | −0.604 | 1.085 | 163.781 | → | 24.114 |
| SPD | 19.025 | 45 | + | 74 | = | 119 | 119.859 | −0.859 | −0.716 | 1.086 | 129.219 | → | 19.025 |
| GRÜNE | 13.453 | 12 | + | 72 | = | 84 | 84.754 | −0.754 | −0.890 | 1.088 | 91.373 | → | 13.453 |
| LINKE | 10.171 | 6 | + | 58 | = | 64 | 64.077 | −0.077 | −0.120 | 1.079 | 69.081 | → | 10.171 |
| CSU | 6.920 | 47 | | – | = | 47 | 43.595 | $\boxed{3.405}$ | $\boxed{7.810}$ | $\boxed{1.000}$ | 47.000 | → | 6.920 |
| SSW | 0.178 | – | | 1 | = | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.207}$ | $\boxed{1.207}$ | → | 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 299 | + | 331 | = | 630 | 630.000 | $\boxed{4.322}$ | $\boxed{18.512}$ | $\boxed{0.207}$ | $\boxed{176.331}$ | → | 100.000 |
| | Using **Absolute** optimization | | | | | | | | | | | | |
| | $\mathbf{q}*100$ | $\mathbf{d}$ | | $\mathbf{a}$ | | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}*100$ |
| | 30 | 31 | | 32 | | 33 | 34 | 35 | 36 | 37 | 38 | | 39 |
| CDU | 26.139 | 143 | + | 21 | = | 164 | 164.678 | −0.678 | −0.412 | 1.083 | $\boxed{177.539}$ | → | 26.139 |
| AfD | 24.114 | 46 | + | 105 | = | 151 | 151.917 | $\boxed{-0.917}$ | −0.604 | 1.085 | 163.781 | → | 24.114 |
| SPD | 19.025 | 45 | + | 74 | = | 119 | 119.859 | −0.859 | −0.716 | 1.086 | 129.219 | → | 19.025 |
| GRÜNE | 13.453 | 12 | + | 72 | = | 84 | 84.754 | −0.754 | −0.890 | 1.088 | 91.373 | → | 13.453 |
| LINKE | 10.171 | 6 | + | 58 | = | 64 | 64.077 | −0.077 | −0.120 | 1.079 | 69.081 | → | 10.171 |
| CSU | 6.920 | 47 | | – | = | 47 | 43.595 | $\boxed{3.405}$ | $\boxed{7.810}$ | $\boxed{1.000}$ | 47.000 | → | 6.920 |
| SSW | 0.178 | – | | 1 | = | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.207}$ | $\boxed{1.207}$ | → | 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 299 | + | 331 | = | 630 | 630.000 | $\boxed{4.322}$ | $\boxed{18.512}$ | $\boxed{0.207}$ | $\boxed{176.331}$ | → | 100.000 |
| | Using **Relative** optimization | | | | | | | | | | | | |
| | $\mathbf{q}*100$ | $\mathbf{d}$ | | $\mathbf{a}$ | | $\mathbf{x}$ | $S*\mathbf{q}$ | $\mathbf{x}-S*\mathbf{q}$ | $(\frac{\mathbf{x}}{S}-\mathbf{q})./\mathbf{q}*100$ | $\mathbf{w}$ | $\mathbf{x}.*\mathbf{w}$ | | $\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}*100$ |
| | 40 | 41 | | 42 | | 43 | 44 | 45 | 46 | 47 | 48 | | 49 |
| CDU | 26.139 | 143 | + | 21 | = | 164 | 164.678 | −0.678 | −0.412 | 1.083 | $\boxed{177.539}$ | → | 26.139 |
| AfD | 24.114 | 46 | + | 105 | = | 151 | 151.917 | $\boxed{-0.917}$ | −0.604 | 1.085 | 163.781 | → | 24.114 |
| SPD | 19.025 | 45 | + | 74 | = | 119 | 119.859 | −0.859 | −0.716 | 1.086 | 129.219 | → | 19.025 |
| GRÜNE | 13.453 | 12 | + | 72 | = | 84 | 84.754 | −0.754 | −0.890 | 1.088 | 91.373 | → | 13.453 |
| LINKE | 10.171 | 6 | + | 58 | = | 64 | 64.077 | −0.077 | −0.120 | 1.079 | 69.081 | → | 10.171 |
| CSU | 6.920 | 47 | | – | = | 47 | 43.595 | $\boxed{3.405}$ | $\boxed{7.810}$ | $\boxed{1.000}$ | 47.000 | → | 6.920 |
| SSW | 0.178 | – | | 1 | = | 1 | 1.120 | −0.120 | $\boxed{-10.702}$ | $\boxed{1.207}$ | $\boxed{1.207}$ | → | 0.178 |
| Sum / $\boxed{\text{Range}}$ | 100.000 | 299 | + | 331 | = | 630 | 630.000 | $\boxed{4.322}$ | $\boxed{18.512}$ | $\boxed{0.207}$ | $\boxed{176.331}$ | → | 100.000 |

$$= \frac{630 * \begin{pmatrix} 0.26139 \\ 0.24114 \\ 0.19025 \\ 0.13453 \\ 0.10171 \\ 0.06920 \\ 0.00178 \end{pmatrix} ./ \begin{pmatrix} 164 \\ 151 \\ 119 \\ 84 \\ 64 \\ 47 \\ 1 \end{pmatrix}}{\min[\text{The resulting vector above}]}$$

$$= \frac{\begin{pmatrix} 1.0041 \\ 1.0061 \\ 1.0072 \\ 1.0090 \\ 1.0012 \\ 0.9276 \\ 0.1214 \end{pmatrix}}{0.9276}$$

$$= \begin{pmatrix} 1.083 \\ 1.085 \\ 1.086 \\ 1.088 \\ 1.079 \\ 1.000 \\ 1.209 \end{pmatrix}.$$

These adjustment vote weights are shown in Columns * 7 of Table 14. The deputies of the most overrepresented party, the CSU, receive one vote each. Others have slightly greater vote weights: the single deputy of the most underrepresented party, the SSW, has the vote weight of 1.207, and the deputies of other Bundestag parties have the vote weights in interval [1.079; 1.088]; so the range of vote powers of most deputies is quite moderate — within 9% only.[8]

As proved in Appendix, Relative optimization is equivalent to minimizing the range of adjustment vote weights; cf. Columns * 6 and * 7 in Tables 6 and 14. Since a greater range of adjustment vote weights means a higher inequality between Bundestag members, the Relative optimization serves *equalizing vote powers of Bundestag members*.

Thus, adjustment vote weights bring the faction powers into precise correspondence with party quotas, making even the tolerance for rounding errors of 0.5 seat unnecessary; see Columns * 9 in Tables 6 and 14. What is likely more important is that they enable to restore the balance between both concepts of representation in the Bundestag with 630 seats, making it more inclusive for local representatives. In fact, it seems better to slightly adjust vote weights for some Bundestag members than to leave them without mandates at all.

## 7 Concluding discussion

Our study would be incomplete without mentioning relevant issues that remain beyond the scope of our analysis. First of all, the 2023/24 electoral reform significantly reformatted the German mixed-member proportional representation system, having strengthened the importance of the second vote (for a party) and weakened the effectiveness of the first vote (for a local representative). It can be said that the reform completed the transformation of the already more important second vote into the first. It is difficult to

[8]Since the SSW faction consists of a single deputy, its deviation from the quota is fixed and behaves as an outlier among large factions' deviations from quotas that are minimized by redistribution of mandates.

judge whether it was an explicit intention or a tolerated side effect of limiting the Bundestag growth at the expense of leaving numerous constituencies without their deputies in the Bundestag. In fact, the reform made the Bundestag less inclusive and upset the balance between the descriptive and agent concepts of representation implemented in the German system. Leaving this political philosophy question about the roles of individual politicians and political institutions open, we nevertheless realize that it is especially acute in times of confrontation between democracy and authoritarianism.

The issue related to the previous one is the way the Bundestag mandates are distributed between the federal state party associations. The German two-tier procedure, which allocates the Bundestag mandates to eligible parties first and then distributes them between local representatives, assumes the dominance of the nationwide institutional approach over the regional personalistic one. The alternative one-tier procedure, when the Bundestag mandates are directly allocated to all federal state associations at once, implies their greater independence, i.e. more political federalism in general. The choice between the two-tier procedure and the one-tier apportionment, which is not only more straightforward but also more accurate, is another question for political discourse.

Another aspect of allocation of Bundestag mandates to federal state party associations is the way their (sub-)quotas are defined. Currently, these sub-quotas are proportional to the votes the parties receive in federal states. Hence, a low turnout in a federal state would imply small Bundestag sub-quotas for all parties, that is, the federal state would receive too few Bundestag mandates, even if the federal state is large. Therefore, it might have sense to define parties' federal state sub-quotas not only in proportion to votes they receive but with taking into account the size of the whole electorate in federal states (even if abstaining from voting), or the number of their citizens including minors, or even the total of their inhabitants; then the voters act as their collective representatives.

The choice of apportionment method to allocate Bundestag mandates is still on the agenda. From 1949 to 1985, the Bundestag seats were distributed using the D'Hondt method, then the Hare/Niemeyer method was used until 2008, so the currently official Sainte-Laguë method is the third in the recent German history. As we have seen, it is not as perfect as is commonly believed. It finds good solutions but not necessarily the best ones. The best and, consequently, most fair apportionments are found using discrete optimization techniques.

Optimization requires rigorous criteria to be fulfilled. By default, the allocation of seats to Bundestag parties assumes minimum deviations from their quotas, which are proportional to the party votes received in elections. These deviations are tolerated within the rounding error of 0.5 seat, but in most cases optimization models can provide a better accuracy. The deviations from quotas can also be measured in relative units like percentage of the quota, which reflects different importance of one seat for small and large party factions. Therefore, the relative optimization criterion can emphasize the minority rights. Another way to control the deviations from quotas is to consider the variance of the deviations. This parameter is a general measure of dispersion in statistics but it is little discussed in the context of apportionment.

Finally, all apportionment bottle-necks can be overcome by introducing adjustment vote weights. The Bundestag can then be restricted to 630 or even 598 seats, include all 299 constituency election winners as direct mandate holders, and ensure that the parties' voting powers would be in exact proportion to the votes they receive in elections. For this purpose it suffices to slightly reduce the vote weights for some direct mandate holders, comparing with that of other Bundestag members, which seems better than to leave them without mandates at all. To minimize the voting power inequalities, the relative optimization model can be applied, which is another argument in its favor.

To conclude, the 2023/24 electoral reform is the 23rd(!) update to the original electoral law of 1949 [Bundestag 2023]. The numerous modifications create certain inconsistencies which in turn require next updates, making the system less and less repairable. In this case, it might be easier to redesign everything from scratch.

# 8 Appendix: Apportionment as a mathematical problems

## 8.1 D'Hondt and Sainte-Laguë apportionment methods

The D'Hondt apportionment method was originally proposed in the United States by Thomas Jefferson as early as 1792. In Europe, it is attributed to the Belgian mathematician and lawyer Victor D'Hondt who reinvented it in 1878 [D'Hondt 1882, D'Hondt 1885, Pukelsheim 2002, D'Hondt method 2025]. This method has numerous mathematical advantages but it is also known for slightly favoring large parties over small ones [Balinski and Young 1979, Lijphart 2003, Schwingenschlögl and Pukelsheim 2006, Pukelsheim 2007, D'Hondt-Verfahren 2025].

The Sainte-Laguë method strives to complete the same task as the D'Hondt method and is very similar to it, being its minor modification. In the USA, it is named after the American statesman Daniel Webster, who proposed it in 1832 for proportional allocation of seats in the United States congressional apportionment [Balinski and Young 1982]. The French mathematician André Sainte-Laguë rediscovered it later and rigorously studied its properties [Sainte-Laguë 1910, Sainte-Laguë method 2025]. In Europe, both methods are referred to as D'Hondt and Sainte-Laguë methods, respectively.

In 1980, the German physicist and electoral expert Hans Schepers, having studied the D'Hondt method used by the German Bundestag, discovered that it disadvantaged smaller parties and suggested an improved version equivalent to the Sainte-Laguë method [Pukelsheim 2002]. At first it was adopted only for certain Bundestag commissions, but since 2009 it has been used to allocate seats both in the German Bundestag and the European Parliament [Sainte-Laguë-Verfahren 2025]. Both the D'Hondt and Sainte-Laguë methods are widely used worldwide, sometimes interchangeably.

The idea of these methods is as follows. The party with the most electoral votes 'purchases' its first parliamentary seat by 'spending' a certain fixed fraction of the total votes it received in the election. At each successive step, the currently 'richest' party acquires a seat, spending a certain fraction of its remaining votes. Thereby, the next seat goes up to the 'highest bidder' — the party with the most votes to spend. In this way, the biggest winners can acquire several seats before a minor party ever gets to make its first 'purchase'. The procedure runs as long as the initial pool of $S$ nominal parliamentary seats is not exhausted.

The only difference between the D'Hondt and the Sainte-Laguë methods is the amount of spending for each purchase. Under the D'Hondt method, the party 'pays' for the first seat an amount that leaves it with only 1/2 of its original number of votes; then for its next seat it pays an amount that leaves it with only 1/3 of its original number of votes, then 1/4, and so on.

Under the Sainte-Laguë method, the party 'pays' for the first seat an amount that leaves it with only 1/3 of its original number of votes; then for its next seat it pays an amount that leaves it with only 1/5 of its original number of votes, then 1/7, and so on. As one can see, the biggest winners 'spend' their votes much faster than under the D'Hondt method, thereby giving way to smaller parties.

Thus, to allocate the next available seat, the algorithm finds the party $i$ with the *largest remainder of votes* in two versions:

$$\text{while } \sum_{i=1}^{n} x_i < S \quad \text{find } i : \max_{i=1,\dots,n} \begin{cases} \underbrace{\left(\dfrac{v_i}{x_i+1}\right)}_{\substack{i\text{th party's}\\ \text{remainder}\\ \text{of votes}}} & \text{under D'Hondt method} \\[2em] \underbrace{\left(\dfrac{v_i}{2x_i+1}\right)}_{\substack{i\text{th party's}\\ \text{remainder}\\ \text{of votes}}} & \text{under Sainte-Laguë method} \end{cases} \quad \Rightarrow \quad x_i = x_i + 1\,, \qquad (2)$$

where

$n$ is the number of parties eligible for parliamentary seats,

$i = 1, \dots, n$ are 'labels' of the parties eligible for parliamentary seats,

$x_i$ is the number of seats that have already been allocated to party $i$ (initially $x_i = 0$),

$S$ is the total number of parliamentary seats to be allocated, and

$v_i$ is the total number of electoral votes that party $i$ received in the elections.

If the given number of seats $S$ is insufficient to guarantee the required accuracy of proportionality to votes then *leveling seats* (also called *adjustment seats*) are added one-by-one until the required accuracy is achieved, i.e. the actual number of seats is increased up to $S+\Delta$, where $\Delta$ is the number of leveling seats. For example, the German Bundestag that until recently had 598 nominal seats was enlarged further to achieve the accuracy of factions' proportionality to party votes to within the rounding error of 0.5 seat.

**The case of minimum seats reserved for certain parties** The situation becomes more complicated when some seats are reserved for certain parties — like a number of seats in the German Bundestag is reserved for parties' local representatives (direct mandate holders). In this case, the allocation of seats takes place in the regular way step-by-step, with the reserved seats being 'redeemed' first. The reserved seats that are 'not redeemed yet' are considered unadjusted but still belonging to the parliament, making its size at the current computation step greater than the number of seats allocated so far through the regular procedure. In 2021, the German Bundestag was allocated under these conditions: with a number of reserved seats and a toleration of three unadjusted seats of the CSU [Tangian 2022c].

## 8.2 Optimization of apportionments

The algorithm (2) does not appear to have any traces of optimization criteria, being purely heuristic. One can only admire the intuition of its inventors who so implicitly implemented the optimization idea. It comes however into play when we speak of minimizing the parliament size or 'maximizing proportionality' of the apportionment.

To formulate rigorously the respective optimization model, let us suppose that $i = 1, \dots, n$ are labels of $n$ parties that, after elections, are entitled to seats in the parliament with $S$ seats. We define the following vectors:[9]

$\mathbf{v} = (v_1, \dots, v_n)'$, — the vote vector, where $v_i > 0$ are the parties' integer-valued numbers of votes received in the election;

$\mathbf{q} = (q_1, \dots, q_n)'$ — the quote vector, where $q_i \geq 0$, $\sum_{i=1}^{n} q_i = 1$, are the parties' fraction-valued quotas for parliamentary seats; basically the quotas are proportional to votes: $q_i = \frac{v_i}{\sum_{i=1}^{n} v_i}$;

$\mathbf{x} = (x_1, \dots, x_n)'$ — the (unknown) apportionment vector, where $x_i \geq 0$, $\sum_{i=1}^{n} x_i = S$, are the parties' integer-valued numbers of parliamentary seats.

[9]In this paper, all vectors are column-vectors.

We measure the apportionment accuracy in absolute and relative terms. The absolute accuracy is the maximum deviation of the parties' shares of parliamentary seats from the quotas, and the relative accuracy is the maximum of these deviations relative to the quotas:

$$\text{Absolute apportionment accuracy} \quad \varepsilon \;=\; \max_i \underbrace{|x_i - Sq_i|}_{\substack{i\text{th faction}\\ \text{fitting error}}}\;, \tag{3}$$

$$\text{Relative apportionment accuracy} \quad \varepsilon \;=\; \max_i \underbrace{\left|\frac{x_i - q_i S}{q_i S}\right|}_{\substack{i\text{th faction}\\ \text{fitting error}\\ \text{relative to}\\ \text{the quota size}}}\;. \tag{4}$$

For every accuracy measure, the apportionment optimization problem is formulated in two versions: (1) given the apportionment accuracy, minimize the parliament size — the task before the 2023/24 German electoral reform, or (2) given the parliament size, minimize the apportionment error — the task after the reform.

## 8.3 Minimizing parliament size $S$ for a given apportionment accuracy $\varepsilon$

Let us consider the first optimization model: Given an apportionment accuracy $\varepsilon$, find the apportionment $\mathbf{x}$ while minimizing the parliament size $S$. In mix-member proportional representation systems this task is often constrained by minimum party seats for local representatives elected in constituencies (direct mandate holders).

**Apportionment accuracy is measured in absolute terms (seats)** The apportionment accuracy constraint in absolute terms (3) means that the $i$th party's share of parliamentary seats $x_i / \sum_{j=1}^n x_j$ is within the party's parliament quota $q_i \pm \varepsilon$:

$$-\varepsilon \le \frac{x_i}{\sum_{j=1}^n x_j} - q_i \le \varepsilon, \quad i = 1,\dots,n. \tag{5}$$

The right inequality in (5) can be rewritten as

$$x_i - (q_i + \varepsilon)\sum_{j=1}^n x_j \le 0, \quad i = 1,\dots,n.$$

In the matrix form, these $n$ inequalities look as follows:

$$\begin{pmatrix} (1-(q_1+\varepsilon))x_1 & -(q_1+\varepsilon)x_2 & \cdots & -(q_1+\varepsilon)x_n \\ -(q_2+\varepsilon)x_1 & (1-(q_2+\varepsilon))x_2 & \cdots & -(q_2+\varepsilon)x_n \\ \cdots & \cdots & \cdots & \cdots \\ -(q_n+\varepsilon)x_1 & -(q_n+\varepsilon)x_2 & \cdots & (1-(q_n+\varepsilon))x_n \end{pmatrix} \le \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

or

$$\underbrace{\begin{pmatrix} 1-(q_1+\varepsilon) & -(q_1+\varepsilon) & \cdots & -(q_1+\varepsilon) \\ -(q_2+\varepsilon) & 1-(q_2+\varepsilon) & \cdots & -(q_2+\varepsilon) \\ \cdots & \cdots & \cdots & \cdots \\ -(q_n+\varepsilon) & -(q_n+\varepsilon) & \cdots & 1-(q_n+\varepsilon) \end{pmatrix}}_{\mathbf{A}_1} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \le \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Similarly, the left inequalities in (5) are represented in the matrix form as well:

$$\underbrace{\begin{pmatrix} -1+(q_1-\varepsilon) & q_1-\varepsilon & \cdots & q_1-\varepsilon \\ q_2-\varepsilon & -1+(q_2-\varepsilon) & \cdots & q_2-\varepsilon \\ \multicolumn{4}{c}{\dotfill} \\ q_n-\varepsilon & q_n-\varepsilon & \cdots & -1+(q_n-\varepsilon) \end{pmatrix}}_{\mathbf{A}_2} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \leq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thereby, we reduce our task to the mixed-integer linear programming problem solvable by function `intlinprog` from the MATLAB Optimization Toolbox:[10]

$$\min_{\mathbf{x}} \underbrace{\mathbf{1}' \cdot \mathbf{x}}_{\substack{\text{size of the} \\ \text{parliament}}} \text{ subject to } \begin{cases} \mathbf{x} \text{ is integer-valued} \\ \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \cdot \mathbf{x} \leq \mathbf{0} & \text{(Constraint: absolute apportionment accuracy)} \\ \mathbf{x} \geq \mathbf{d} & \text{(Constraint: minimum number of party seats)} \end{cases} \tag{6}$$

where

$\mathbf{1}$ is the $n$-vector of 1s,

$\mathbf{x}$ is the apportionment $n$-vector of seats allocated to $n$ parties,

$\mathbf{d}$ is the non-negative integer-valued $n$-vector of parties' direct mandates, e.g. $\mathbf{d} = \{100, 80, \ldots, 0\}'$,

$\mathbf{0}$ is the $2n$-vector of 0s.

**Apportionment accuracy is measured in relative terms (fraction of the quota)** The apportionment accuracy constraint in relative terms (4) implies the inequalities

$$-\varepsilon \leq \frac{\frac{x_i}{\sum_{j=1}^n x_j} - q_i}{q_i} \leq \varepsilon, \quad i = 1, \ldots, n. \tag{7}$$

The right inequalities in (7) can be rewritten as

$$x_i - (q_i + \varepsilon q_i) \sum_{j=1}^n x_j \leq 0, \quad i = 1, \ldots, n,$$

which by analogy with the previous case is equivalent to

$$\underbrace{\begin{pmatrix} 1-(q_1+\varepsilon q_1) & -(q_1+\varepsilon q_1) & \cdots & -(q_1+\varepsilon q_1) \\ -(q_2+\varepsilon q_2) & 1-(q_2+\varepsilon q_2) & \cdots & -(q_2+\varepsilon q_2) \\ \multicolumn{4}{c}{\dotfill} \\ -(q_n+\varepsilon q_n) & -(q_n+\varepsilon q_n) & \cdots & 1-(q_n+\varepsilon q_n) \end{pmatrix}}_{\mathbf{R}_1} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \leq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

[10]Matrices $\mathbf{A}_1, \mathbf{A}_2$ can be generated using the MATLAB function `diag(x)` to create a diagonal matrix with vector $\mathbf{x}$ at its diagonal and the operation of adding a scalar to all elements of a vector or matrix:

$$\begin{aligned} \mathbf{A}_1 &= \mathtt{diag}(\mathbf{1}) - \mathtt{diag}(\mathbf{q}+\varepsilon) \cdot \mathbf{1} \\ \mathbf{A}_2 &= -\mathtt{diag}(\mathbf{1}) + \mathtt{diag}(\mathbf{q}-\varepsilon) \cdot \mathbf{1}, \end{aligned}$$

where $\mathbf{1}$ and $\mathbf{1}$ are the $n$-vector and $n \times n$-matrix of 1s, respectively.

The left inequalities in (7) are represented in the matrix form as well:

$$\underbrace{\begin{pmatrix} -1+(q_1-\varepsilon q_1) & q_1-\varepsilon q_1 & \cdots & q_1-\varepsilon q_1 \\ q_2-\varepsilon q_2 & -1+(q_2-\varepsilon q_2) & \cdots & q_2-\varepsilon q_2 \\ \dots & \dots & \dots & \dots \\ q_n-\varepsilon q_n & q_n-\varepsilon q_n & \cdots & -1+(q_n-\varepsilon q_n) \end{pmatrix}}_{\mathbf{R}_2} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \le \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The task is thereby reduced to the mixed-integer linear programming problem also solvable by function `intlinprog` from the MATLAB Optimization Toolbox:[11]

$$\min_{\mathbf{x}} \underbrace{\mathbf{1}'\cdot\mathbf{x}}_{\substack{\text{size of the}\\ \text{parliament}}} \quad \text{subject to} \begin{cases} \mathbf{x} \ \text{ is integer-valued} & \\ \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{pmatrix} \cdot \mathbf{x} \le \mathbf{0} & \text{(Constraint: relative apportionment accuracy)} \\ \mathbf{x} \ge \mathbf{d} & \text{(Constraint: minimum number of party seats)} \end{cases}$$

### 8.4 Minimizing the apportionment errors $\varepsilon$ for a given parliament size $S$

Now we consider the second optimization model: Given a parliament size $S$, find the apportionment $\mathbf{x}$ that minimizes the apportionment errors

**Minimizing absolute apportionment errors** In this case, we minimize the deviations from party quotas $|x_i - Sq_i|$ expressed in seats, or the same expression divided by the constant $S$. We come to the optimization problem

$$\min_{\mathbf{x}} \underbrace{\max_i \underbrace{\left|\frac{x_i}{S} - q_i\right|}_{\substack{i\text{th faction}\\ \text{absolute}\\ \text{fitting error}\\ \text{divided by } S}}}_{\substack{\text{absolute}\\ \text{apportionment}\\ \text{accuracy}\\ \text{divided by } S}} \quad \text{subject to} \begin{cases} \mathbf{x} \ \text{ is integer-valued} \\ \mathbf{x} \ge \mathbf{d} \quad \text{(Constraint: minimum number of party seats)} \end{cases} \tag{8}$$

Since the number of seats $S$ is fixed, the only way to reduce the absolute apportionment accuracy threshold in (8) is to move seats from one party faction to another. This observation suggests the following directed search:

1. Find the party faction $j$ with the greatest deviation from its quota, i.e.

$$j : \left|\frac{x_j}{S} - q_j\right| = \max_i \left|\frac{x_i}{S} - q_i\right| .$$

2. If the $j$th party quota is exceeded, i.e. $\frac{x_j}{S} > q_j$, then move one its seat to another party faction, trying to maximally reduce $\max_i \left|\frac{x_i}{S} - q_i\right|$; test all party factions because for small factions adding even one seat can significantly change the deviation from the quota.

[11] Matrices $\mathbf{R}_1, \mathbf{R}_2$ can be generated in the same way as described in the previous footnote:

$$\begin{aligned} \mathbf{R}_1 &= \mathtt{diag}(\mathbf{1}) - \mathtt{diag}(\mathbf{q}+\varepsilon\mathbf{q})\cdot\mathbf{1} \\ \mathbf{R}_2 &= -\mathtt{diag}(\mathbf{1}) + \mathtt{diag}(\mathbf{q}-\varepsilon\mathbf{q})\cdot\mathbf{1}\,. \end{aligned}$$

3. If the party quota is not filled, i.e. $\frac{x_j}{S} < q_j$, then increase $x_j$ by moving one seat from some other faction, trying to maximally reduce $\max_i \left|\frac{x_i}{S} - q_i\right|$; test all party factions.

4. If no such seat exchange improves the absolute apportionment accuracy then the current apportionment is indeed optimal.

5. Otherwise, repeat the cycle as long as the absolute apportionment accuracy can be improved.

**Minimizing relative apportionment errors using adjustment vote weights** To find the apportionment of $S$ parliamentary seats with the maximum relative accuracy, we have to solve the optimization problem (8), replacing $\dfrac{x_i}{S} - q_i$ by $\dfrac{\frac{x_i}{S} - q_i}{q_i}$ :

$$\underbrace{\min_{\mathbf{x}} \max_i \underbrace{\left|\frac{x_i}{q_i S} - 1\right|}_{\substack{i\text{th faction}\\ \text{relative}\\ \text{fitting error}}}}_{\substack{\text{relative}\\ \text{apportionment}\\ \text{accuracy}}} \quad \text{subject to} \begin{cases} \mathbf{x} \ \text{is integer-valued} \\ \mathbf{x} \geq \mathbf{d} \quad \text{(Constraint: minimum number of party seats)} \end{cases} \tag{9}$$

Solving (9) is less straightforward because one seat has different significance for large and small factions. To overcome this complication, we consider *adjustment vote weights*, with which the voting power of party factions is brought into exact accordance with their parliamentary quotas. For instance, if a parliament has 99 seats and two party factions have equal quotas $\mathbf{q} = \{0.5, 0.5\}'$, the only possible apportionment $\mathbf{x} = \{49, 50\}'$ results in underrepresentation of one party and overrepresentation of another. To ensure parity, the vote weight of the members of the underrepresented faction is made 50/49, which equalizes the factions' voting power.

Thus, if $q_i$ is the quota of the $i$th party then the adjustment vote weight $w_i$ has to satisfy the equation

$$w_i x_i = S q_i \quad \Leftrightarrow \quad w_i = S q_i / x_i \ , \quad i = 1, \ldots, n \ . \tag{10}$$

The vector form of these equations is as follows:[12]

$$\mathbf{w} = S\mathbf{q}./\mathbf{x} \quad \text{(the adjustment vote weight vector)}\ .^{13}$$

In our model, the members of the most overrepresented faction (with the smallest adjustment vote weight) are assigned the vote weight = 1, and the adjustment vote weights for the members of other factions are obviously $> 1$, i.e. we *redefine*

$$\mathbf{w} = \frac{S\mathbf{q}./\mathbf{x}}{\min[S\mathbf{q}./\mathbf{x}]} \quad \text{(the normalized adjustment vote weight vector)}\ . \tag{11}$$

The following obvious statement enables improving the relative apportionment accuracy by equalizing adjustment vote weights.

**Proposition 1 (Improving relative apportionment accuracy by equalizing adjustment vote weights)** *While improving the relative apportionment accuracy (= minimizing $\varepsilon$), the range of adjustment vote weights* Range($\mathbf{w}$) *is minimized, and vice versa.*

[12]The following operation ./ is the element-by-element division of vectors, e.g. $(25,16)./(5,2) = (5,8)$. The operation .* (used below) is the element-by-element multiplication of vectors, e.g. (25,16).*(5,2)=(125,32).

[13]The notation like min[$\mathbf{x}$] means the corresponding operation on the elements of the column vector in the brackets.

*Proof.* By definition,

$$\begin{aligned} \text{Vector of relative deviations from quotas} &= (\mathbf{x} - S\mathbf{q})./(S\mathbf{q}) \\ &= \mathbf{x}./(S\mathbf{q}) - \mathbf{1} \ . \end{aligned}$$

The minimization of relative deviations from quotas means

$$\mathbf{x}./(S\mathbf{q}) - \mathbf{1} \to \mathbf{0} \quad \Leftrightarrow \quad \mathbf{x} \to S\mathbf{q} \quad \Leftrightarrow \quad \mathbf{w} = \frac{S\mathbf{q}./\mathbf{x}}{\min[S\mathbf{q}./\mathbf{x}]} \to \mathbf{1} \quad \Leftrightarrow \quad \text{Range}(\mathbf{w}) \to 0 \ .$$

QED

The search for the best apportionment regarding the relative accuracy measure follows Algorithm 1–5 from the previous paragraph. The only difference is that the largest relative deviations from the quota are recognized from the adjustment vote weights: the greatest indicates the party faction to increase, and the least — 1 — indicates the party faction to reduce.

Thus, the optimization problem of minimizing *relative* deviations of factions from quotas is reformulated in terms of minimizing the range of adjustment vote weights. Since a greater range of adjustment vote weights means a higher inequality of members of parliament, minimizing *relative* deviations from quotas means *equalizing individual powers* of the members of parliament.

## 8.5 Universality of Algorithm 1–5 and lexicographic optimization

Algorithm 1–5 suggested above can be used under various constraints and even with an option of starting from an arbitrary apportionment.

This is important in the *lexicographic optimization* when two optimization criteria — absolute and relative — are successively applied. For example, there may be several optimal apportionments with respect to the absolute accuracy criterion, and then we apply the relative optimization to select among them the best one regarding relative accuracy. Conversely, a non-unique optimal apportionment with respect to the relative accuracy criterion can be further improved by using the absolute accuracy criterion.

The additional advantage is the possibility to add or remove parliamentary seats one-by-one, i.e. gradually increase/decrease the size of the parliament, controlling the accuracy of the resulting *always optimal* apportionment. Using such a progression, one can solve the optimization problem from Section 8.3 — given apportionment accuracy, find minimum parliament size $S$. For this purpose, the parliament is increased until the required apportionment accuracy is achieved.

## 8.6 Computer implementation

We will compare four apportionment methods: that of D'Hondt (DH), Sainte-Laguë (SL), Absolute Optimization (AO), and Relative Optimization (RO). For this purpose, we define function $f$, which to vector arguments $\mathbf{q}$ (quotas), $\mathbf{d}$ (direct mandates), 'Method' (of apportionment) and $S$ (size of parliament) puts into correspondence the apportionment vector

$$\mathbf{x} = f(\mathbf{q}, \mathbf{d}, \text{Method}, S) \ , \qquad \text{Method = DH, SL, AO, RO,} \qquad S = S_{\min} \ , \ S_{\min} + 1, \ \ldots \ , \ S_{\max} \ . \qquad (12)$$

For Methods DH and SL, $f$ is computed by algorithm (2), and for Methods AO and RO, we use Algorithm 1–5. Both methods AO and RO apply the two-level lexicographic optimization as described in Section 8.5; for example, if the absolute accuracy criterion results in several optimal solutions, the best one among them is selected with respect to the relative accuracy criterion.

The apportionment $\mathbf{x}$ is used to derive further $n$-vectors, which constitute the tables below:

$\mathbf{q} * 100$ — vector of party quotas, in %,

$\mathbf{a}$ — vector of adjustment (leveling) seats,

$S * \mathbf{q}$ — vector of party quotas expressed in (fractional) number of parliamentary seats,

$\mathbf{x} - S * \mathbf{q}$ — vector of absolute deviations from the quotas, in number of seats; cf. with (8),

$\left(\frac{\mathbf{x}}{S} - \mathbf{q}\right) ./\mathbf{q} * 100$ — vector of relative deviations from the quotas, in %; cf. with (9),

$\mathbf{w} = \frac{S\mathbf{q}./\mathbf{x}}{\min[S\mathbf{q}./\mathbf{x}]}$ — vector of faction members' adjustment vote weights normalized, that is, with the minimum adjustment vote weight = 1,

$\mathbf{x}. * \mathbf{w}$ — vector of faction's voting power, that is, the total of the votes with adjustment weights,

$\frac{\mathbf{x}. * \mathbf{w}}{\mathbf{x}'\mathbf{w}} * 100$ — vector of faction's vote power, in %, which is brought in the exact accordance with the quotas.

The computer program is written in MATLAB. It outputs a head LaTeX file that calls the computed LaTeX tables and figures saved as eps-files. The head file is used as a LaTeX template for this paper.

### 8.7 Visualizing the performance of apportionment methods

As follows from Tables 6, 9 and 12, the Sainte-Laguë method, which is currently used to allocate the Bundestag seats to parties, is not optimal. It finds apportionments that correspond well to party quotas but not necessarily the best ones. Table 6 demonstrates that even the D'Hondt method gives a better result, not to mention the optimization models. On the other hand, Tables 9 and 12 show that the Sainte-Laguë method can be superior to the D'Hondt method. Now we trace the performance of four apportionment methods for the Bundestag of variable size to see which methods are more accurate on the average and, consequently, 'more fair'. The 'more fair' means that deviations of party faction from quotas, either *absolute* or *relative*, are smaller.

**Apportionment methods from the viewpoint of the absolute accuracy criterion.** Figure 1 visualizes the *absolute* accuracy (i.e. errors are measured in seats) of apportionments $\mathbf{x}_{\mathrm{SL}}$, $\mathbf{x}_{\mathrm{DH}}$, $\mathbf{x}_{\mathrm{AO}}$ and $\mathbf{x}_{\mathrm{RO}}$[14] of the Bundestag with $S = 550, \ldots, 720$ seats and no direct mandates ($\mathbf{d} = \mathbf{0}$) for party quotas $\mathbf{q}$ from Table 6. The four upper curves plot maximum *positive* deviations from the quotas for the four apportionments, and the lower curves plot maximum *negative* deviations.[15] The curve color indicates the method of apportionment as in the figure's text descriptions, and the horizontal yellow stripe — the tolerated apportionment inaccuracies of $\pm 0.5$ seat.

Using the notation of Section 8.6, we plot the following eight functions:[16]

$$\left.\begin{array}{l} \max[\mathbf{x}_{\mathrm{SL}} - S\mathbf{q}] \\ \min[\mathbf{x}_{\mathrm{SL}} - S\mathbf{q}] \\ \max[\mathbf{x}_{\mathrm{DH}} - S\mathbf{q}] \\ \min[\mathbf{x}_{\mathrm{DH}} - S\mathbf{q}] \\ \max[\mathbf{x}_{\mathrm{AO}} - S\mathbf{q}] \\ \min[\mathbf{x}_{\mathrm{AO}} - S\mathbf{q}] \\ \max[\mathbf{x}_{\mathrm{RO}} - S\mathbf{q}] \\ \min[\mathbf{x}_{\mathrm{RO}} - S\mathbf{q}] \end{array}\right. \qquad S = 550, \ldots, 720\,.$$

[14]SL stands for the Sainte-Laguë method, DH — for the D'Hondt method, AO — for the Absolute Optimization, and RO — for the Relative Optimization.

[15]The avoid overlapping, the curves are slightly shifted vertically resulting in minor visual inaccuracies.

[16]$\max[\mathbf{x}]$ for vector $\mathbf{x} = \{x_1, \ldots, x_n\}$ means $\max(x_1, \ldots, x_n)$.

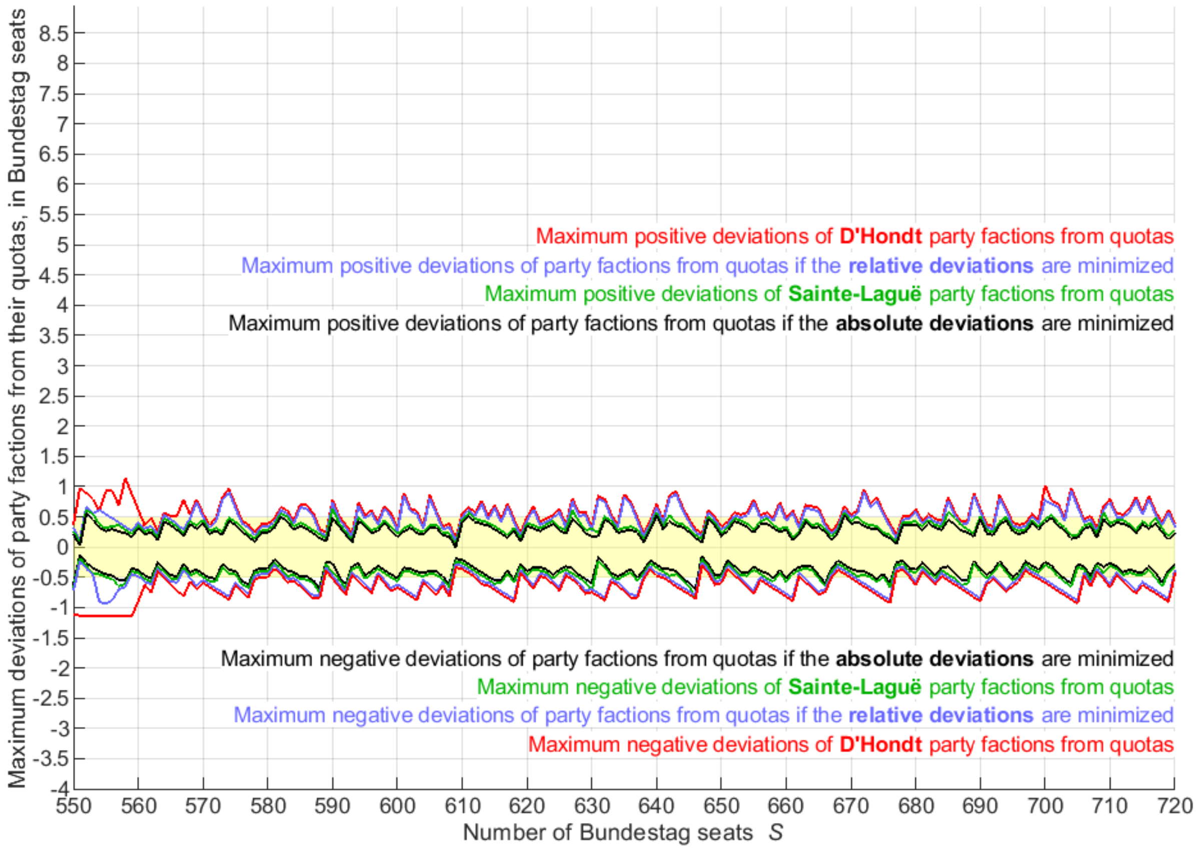


Figure 1: Maximum deviations of party factions from their quotas for the Bundestag with **no direct mandates**. The yellow zone shows the tolerated deviations of factions from quotas within 0.5 seat.

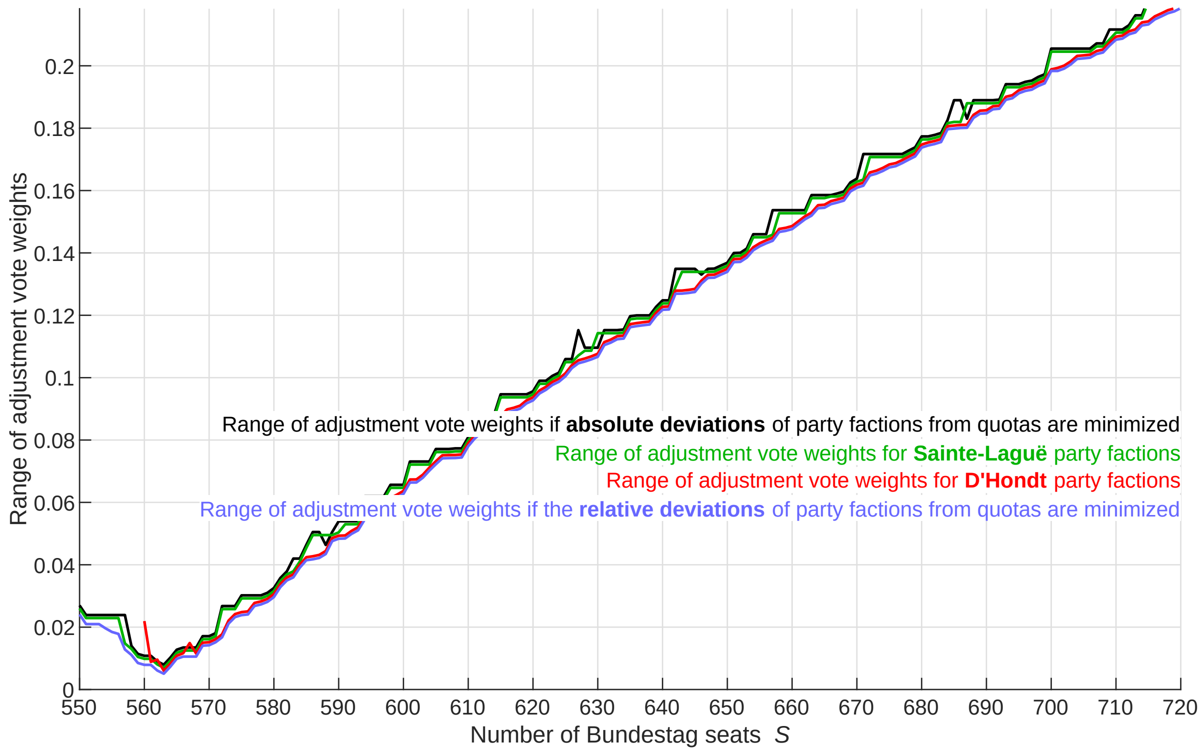


Figure 2: Range of adjustment vote weights for the Bundestag with **no direct mandates**.

The values of these functions for the Bundestag with $S = 630$ seats are highlighted by frames in Columns * 5 of Table 6: 0.405 and $-0.678$ for the Sainte-Laguë method and 0.322 and $-0.595$ for other three methods.

The curves in Figure 1 have a sawtooth shape because every seat added to the Bundestag changes the proportion between party factions as well as replaces the party with the greatest deviation from quota, which implies a positive or negative leap of the maximum apportionment error. The rather chaotic behavior of the curves is reflected by their relatively low correlation in Table 15.

Regardless of the erratic behavior, the black curves of Absolute Optimization run within the yellow stripe, i.e., maximum apportionment errors are within the tolerated $\pm 0.5$ seat. The green Sainte-Laguë curves run very close to the black Absolute Optimization curves. In contrast to that, the red D'Hondt curves seldom enter the yellow stripe and run almost synchronously with the blue Relative Optimization curves.

It is not surprising that Figure 1 demonstrates the Absolute Optimization's superiority to other three methods with respect to the absolute accuracy criterion. What is less obvious is that the Sainte-Laguë method looks so much superior to the D'Hondt method — so that the failure of the former in Table 6 should be considered occasional.

**Apportionment methods from the viewpoint of the relative accuracy criterion.** By virtue of Proposition 1, the relative accuracy of apportionment $\mathbf{x}$ (i.e. errors are measured in % of the quotas) is 1–1 linked to the range of the respective adjustment vote weights $\mathbf{w}$. Therefore, to visualize the relative accuracy of apportionments calculated using the four methods, it suffices to plot the following four functions (for the notation see Section 8.6):

$$\begin{aligned}
\operatorname{range}(\mathbf{w}_{\mathrm{SL}}) &= \operatorname{range}\left(\frac{S\mathbf{q}./\mathbf{x}_{\mathrm{SL}}}{\min[S\mathbf{q}./\mathbf{x}_{\mathrm{SL}}]}\right), \\
\operatorname{range}(\mathbf{w}_{\mathrm{DH}}) &= \operatorname{range}\left(\frac{S\mathbf{q}./\mathbf{x}_{\mathrm{DH}}}{\min[S\mathbf{q}./\mathbf{x}_{\mathrm{DH}}]}\right), \\
\operatorname{range}(\mathbf{w}_{\mathrm{AO}}) &= \operatorname{range}\left(\frac{S\mathbf{q}./\mathbf{x}_{\mathrm{AO}}}{\min[S\mathbf{q}./\mathbf{x}_{\mathrm{AO}}]}\right), \\
\operatorname{range}(\mathbf{w}_{\mathrm{RO}}) &= \operatorname{range}\left(\frac{S\mathbf{q}./\mathbf{x}_{\mathrm{RO}}}{\min[S\mathbf{q}./\mathbf{x}_{\mathrm{RO}}]}\right),
\end{aligned}$$

where $\mathbf{x}_{\mathrm{SL}}$, $\mathbf{x}_{\mathrm{DH}}$, $\mathbf{x}_{\mathrm{AO}}$ and $\mathbf{x}_{\mathrm{RO}}$ are apportionments of the Bundestag with $S = 550, \ldots, 720$ seats and no direct mandates ($\mathbf{d} = \mathbf{0}$) for party quotas $\mathbf{q}$ from Table 6. The values of these functions for the Bundestag with $S = 630$ seats are highlighted by frames at the bottom of Columns * 7: 0.130 for the Sainte-Laguë method and 0.123 for other three methods.

The four curves are displayed in Figure 2. They are colored as indicated in the figure's text description, exactly as in Figure 1. As one could expect, the lowest blue curve is that of the Relative Optimization, proving that this model surpasses other methods with respect to the relative accuracy, whereas the Absolute Optimization shows the worst relative accuracy. The smoothness of these curves differ from that in Figure 1: here, the blue Relative Optimization curve is more smooth than the black Absolute Optimization curve; in Figure 1 it is vice versa. The behavior of the Sainte-Laguë and D'Hondt curves are similar to their behavior in Figure 1: the former mostly coincides with the Absolute Optimization curve, whereas the latter — with the Relative Optimization curve.

All four curves have minimum at the Bundestag size of $S = 563$ seats, when one SSW mandate best corresponds to the SSW quota $q_{\mathrm{SSW}}$ (see Table 2), satisfying the equation:

$$S \cdot q_{\mathrm{SSW}} = 1 \quad \Leftrightarrow \quad S = q_{\mathrm{SSW}}^{-1} = \frac{42833356}{76138} \approx 563 \, .$$

The red curve of the D'Hondt method enters the plot shortly before the minimum, at the Bundestag size of $S = 560$, when the D'Hondt method allocates the first seat to the SSW. As long as the SSW has no seat, its relative deviation from quota is equal to $-100\%$, implying the adjustment vote weight to be infinite and not shown in the plot. In a sense, the D'Hondt method delays in allocating the first seat to the SSW, because the Bundestag with $S = 282$ seats suffices to accommodate a SSW mandate holder with a tolerated apportionment inaccuracy of $-0.5$ seat. Indeed, it must hold

$$S \cdot q_{\text{SSW}} > 1 - 0.5 \quad \Leftrightarrow \quad S > 0.5 \cdot q_{\text{SSW}}^{-1} = 0.5 \cdot \frac{42833356}{76138} = 281.29 \; .$$

Here again, we observe the 'discrimination' of weak parties by the D'Hondt method.

The steady growth of the four curves after their minimum at $S = 563$ is caused by the particularity of the SSW faction with a single mandate. Due to its small quota, the SSW will receive its second mandate no earlier than when the Bundestag reaches the size of $S = 844$ seats. Indeed, the relative deviation of one mandate from the SSW quota $(1 - Sq_{\text{SSW}})/(Sq_{\text{SSW}})$ is negative and decreases as long as its absolute value is smaller than the positive deviation of two mandates from the SSW quota $(2 - Sq_{\text{SSW}})/(Sq_{\text{SSW}})$, then the SSW relative deviation from quota switches from the negative to positive value, i.e., it must hold

$$\frac{|\overbrace{1 - Sq_{\text{SSW}}}^{<0}|}{Sq_{\text{SSW}}} \leq \frac{2 - Sq_{\text{SSW}}}{Sq_{\text{SSW}}} \quad \Leftrightarrow \quad Sq_{\text{SSW}} - 1 \leq 2 - Sq_{\text{SSW}} \quad \Leftrightarrow \quad S \leq 1.5 \cdot q_{\text{SSW}}^{-1} = 1.5 \cdot \frac{42833356}{76138} \approx 844 \; .$$

As the Bundestag grows, the SSW is becoming more and more underrepresented in proportion to its quota, which is compensated by an increase in its adjustment vote weight. Since the SSW faction consists of a single member, the negative deviation from the quota increases very significantly — at least till $-50\%$. Due to larger quotas, other party factions receive mandates more frequently and their relative deviations from quotas are not that significant, resulting in the dominance of SSW adjustment vote weight over that of other parties. All of these explain why the adjustment vote range growth is determined by the SSW.

**Accuracy of the Bundestag apportionment for major parties.** Thus, the dominance of the SSW adjustment vote weight over that of other parties is caused by the exceptional status of the SSW, whose very small quota significantly distorts the general picture. Let us see how a more regular situation could look like. For this purpose, we consider the Bundestag without the SSW, retaining all other elements of our models.

The result of this reduction is displayed in Figures 3 and 4. If Figures 1 and 3 are no different, then the same cannot be said about Figures 2 and 4. Without the exceptional dominance of the SSW adjustment vote weight, the curves in Figure 4 have neither global minimum, nor an increasing trend. The range of adjustment vote weights for the four curves in Figure 4 is $\leq 0.024$ (corresponding to the relative inaccuracy $\leq 2.10\%$ of the quotas). As for the relative optimization (the blue curve), the accuracy is even higher: the range of adjustment vote weights is $\leq 0.019$ (corresponding to the relative inaccuracy $\leq 1.30\%$ of the quotas).

**Apportionment of the Bundestag with 276 direct mandates.** We continue to analyze allocations of Bundestag seats to major parties, i.e. without the SSW, but now with the 276 constituency election winners as direct mandate holders. Figures 5 and 6 characterize the absolute and relative accuracy of the apportionments calculated using the four methods for a variable Bundestag size. Since the SSW mandate is excluded, the actual distribution of seats corresponds to the case of the Bundestag with $S = 629$ seats instead of 630. Figure 5, which characterizes the absolute apportionment accuracy, shows that the required accuracy to within 0.5 seat is achieved exactly at $S = 629$. After this point, all the 276 direct mandates are appropriately adjusted, and the apportionment accuracy behaves exactly as in Figure 3.

Figure 6 demonstrates the same trend but from the viewpoint of relative accuracy. As long as the direct mandates are not adjusted, the four curves go down, meaning that the relative accuracy of the Bundestag apportionment improves as its size increases. After the direct mandates have been adjusted at $S = 629$ seats, the four curves behave exactly as in Figure 4.

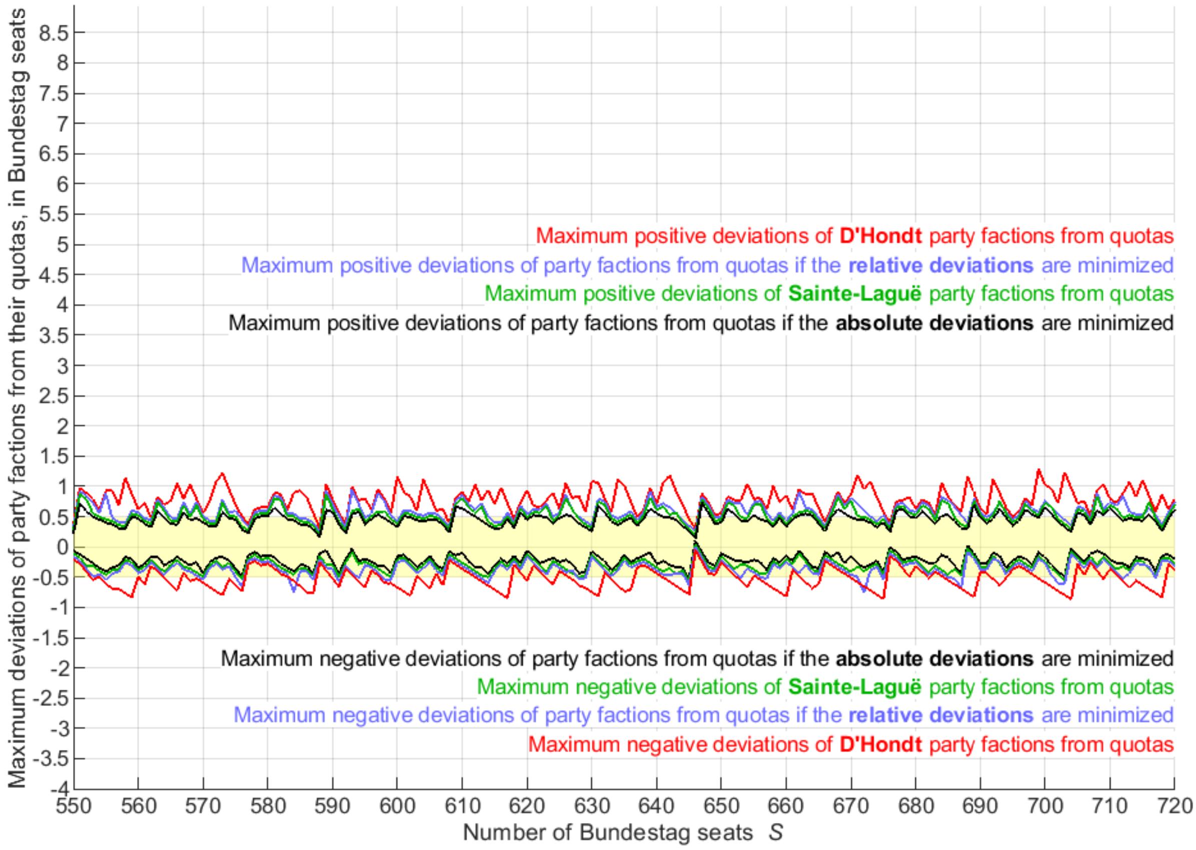


Figure 3: Maximum deviations of party factions from their quotas for the Bundestag with **no direct mandates and no SSW**. The yellow zone shows the tolerated deviations of factions from quotas within 0.5 seat.

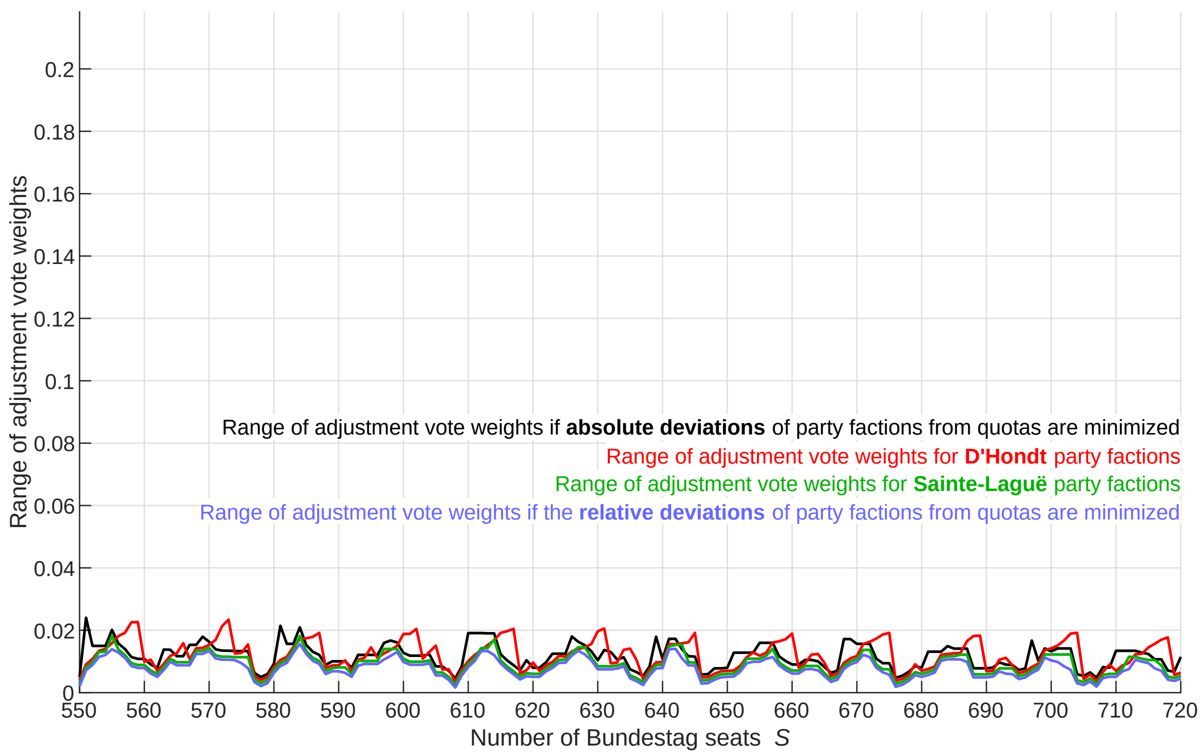


Figure 4: Range of adjustment vote weights for the Bundestag with **no direct mandates and no SSW**.

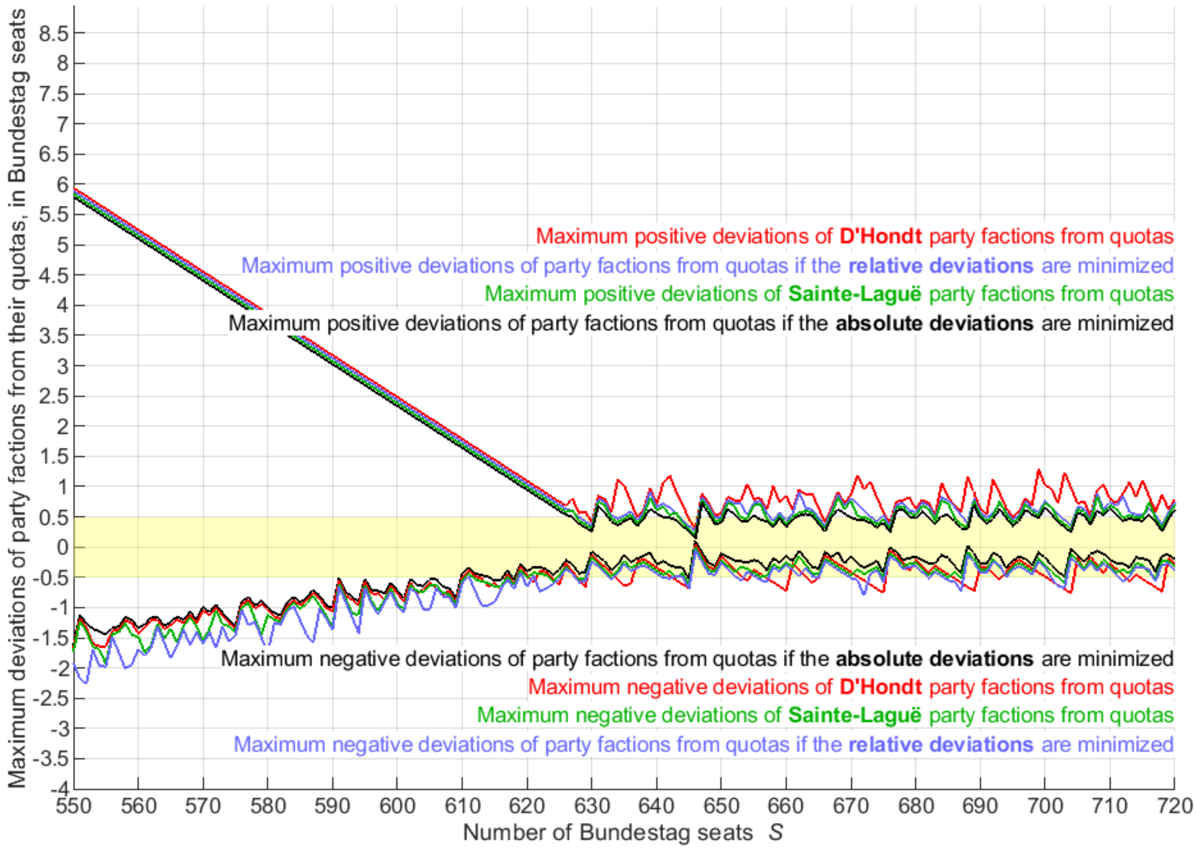


Figure 5: Maximum deviations of party factions from their quotas for the Bundestag with **276 direct mandates and no SSW**. The yellow zone shows the tolerated apportionment inaccuracy within 0.5 seat.

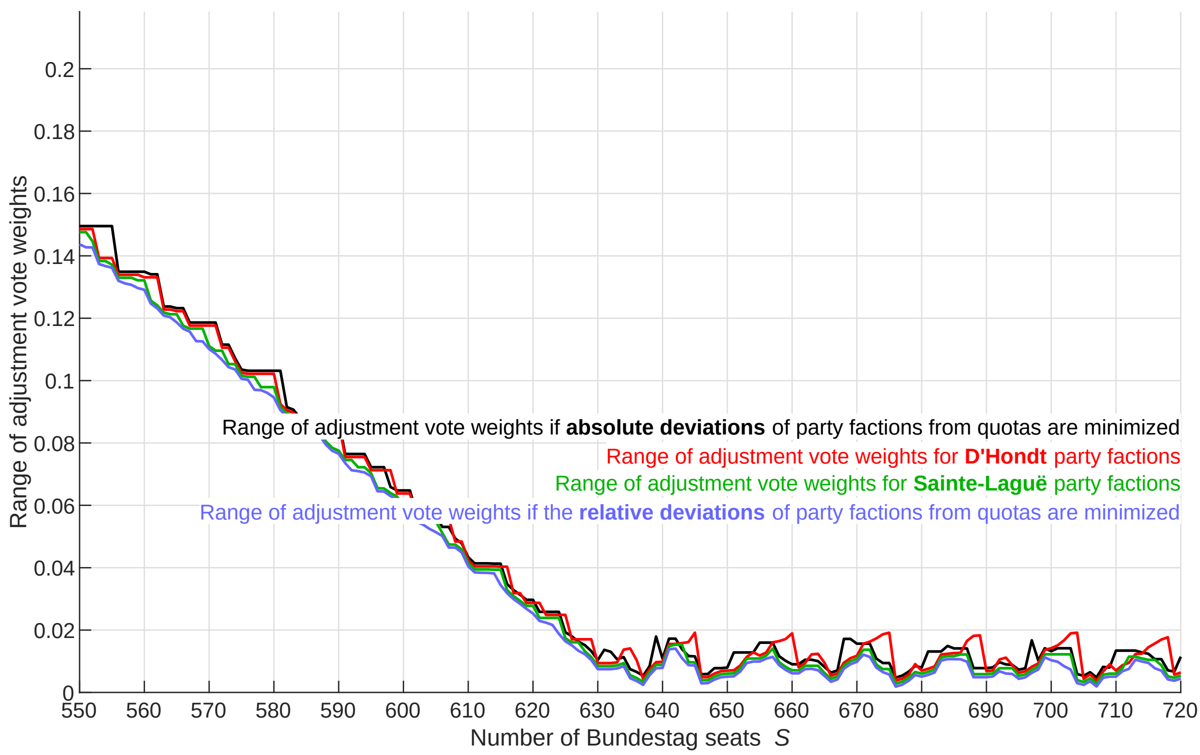


Figure 6: Range of adjustment vote weights for the Bundestag with **276 direct mandates and no SSW**.

**Accuracy of apportionment of the Bundestag with 299 direct mandates.** Figures 7 and 8 trace the apportionment accuracy for the growing Bundestag with *all* 299 constituency election winners as direct mandate holders. The only difference from the case of 276 mandates is that the 299 direct mandates need more seats to achieve the apportionment accuracy of $\pm 0.5$ seat. As follows from (1) and Figure 7, this is fulfilled for the Bundestag size of $S = 672$ seats.

If the Bundestag has fewer than 672 seats then the apportionment inaccuracies exceed 0.5 seat, which can be compensated by adjustment vote weights discussed in Section 6. As follows from Table 14 and Figure 8, the adjustment vote weights of the major parties (except for the SSW) differ not much: from 1.000 for the CSU as the most overrepresented party, to 1.088 for GRÜNE. That is, the inequality of vote powers of the major Bundestag parties is smaller than 9%. The SSW with its single mandate gets the adjustment vote weight of 1.207, but this exception does not affect the balance of powers in the Bundestag.

Figure 8 allows for similar estimations for an arbitrary Bundestag size and an arbitrary configuration of direct mandates. For the former 598 regular Bundestag seats and the given 299 constituency election winners as direct mandate holders, the inaccuracies of the four apportionments calculated using our four methods (from ca. $-2$ to 5.5 seats; see Figure 7) can be compensated by the adjustment vote weights with the range of about 13% (see Figure 8), meaning that even in this case the voting powers of Bundestag members are quite comparable.

**Interdependence of the curves in Figures 1–8.** Tables 15–18 show the correlations between the curves in Figures 1–2, 3–4, 5–6 and 7–8. Regarding the absolute accuracy, the Sainte-Laguë method performs very similarly to the Absolute Optimization, whose 'upper' and 'lower' curves for positive and negative deviations from quotas, respectively, are highly correlated. The same holds for the D'Hondt method and the Relative Optimization, whose 'upper' and 'lower' curves also correlate significantly though not that strongly.

As for the relative accuracy, the four curves of the adjustment vote weight range highly correlate with each other but not with the curves that characterize the absolute accuracy. This means that the absolute and relative accuracy criteria are not so much interchangeable as they are complementary.

Table 16, being dedicated to the Bundestag apportionment accuracy for major parties without direct mandates, exhibits lower correlations because of lack of joint increasing/decreasing trends inherent in the curves of other coupled figures. Generally speaking, the more salient the joint trend, the higher the correlation between the curves in the tables.

**The Sainte-Laguë and D'Hondt methods from an optimization perspective.** Summing up what has been observed, we can make the following informal conclusion:

*The performance of the Sainte-Laguë method approaches Absolute Optimization. The performance of the D'Hondt method is less accurate than that of the Sainte-Laguë method and leans toward Relative Optimization.*

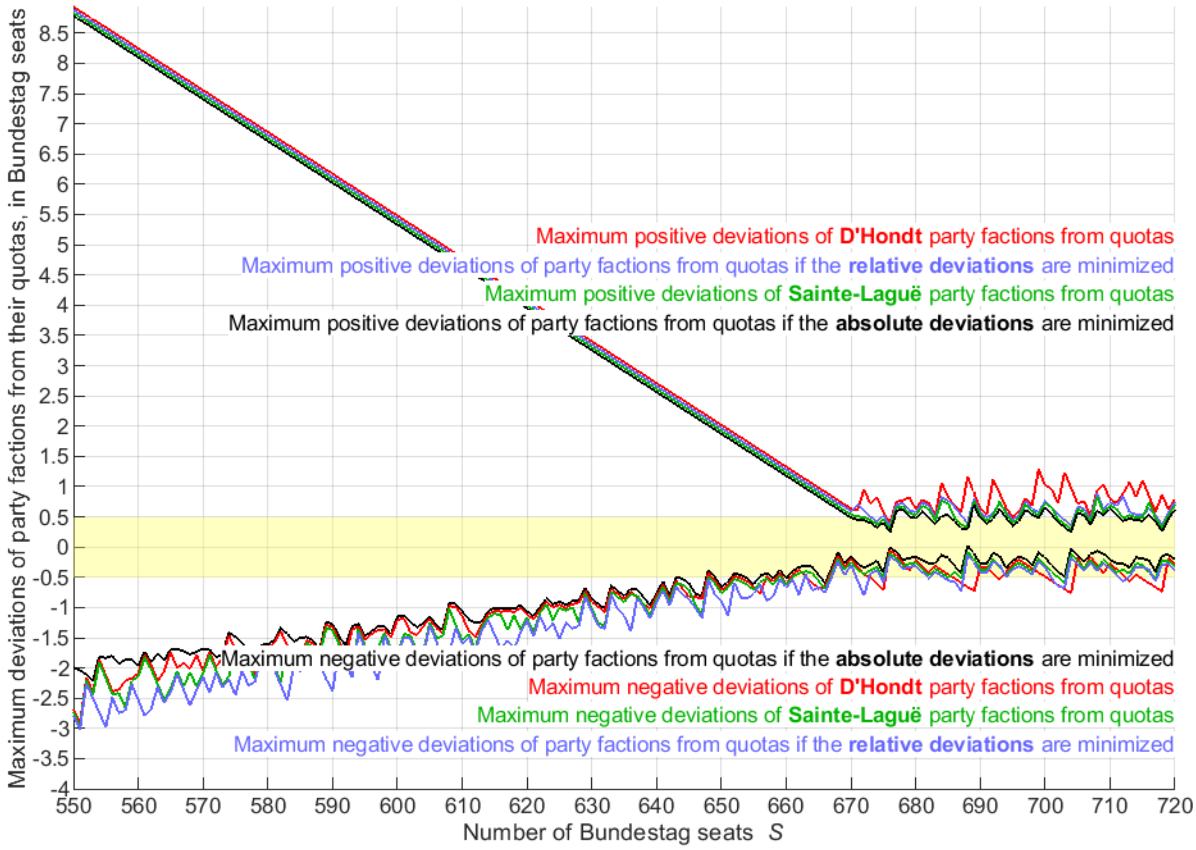


Figure 7: Maximum deviations of party factions from their quotas for the Bundestag with **299 direct mandates and no SSW**. The yellow zone shows the tolerated apportionment inaccuracy within 0.5 seat.

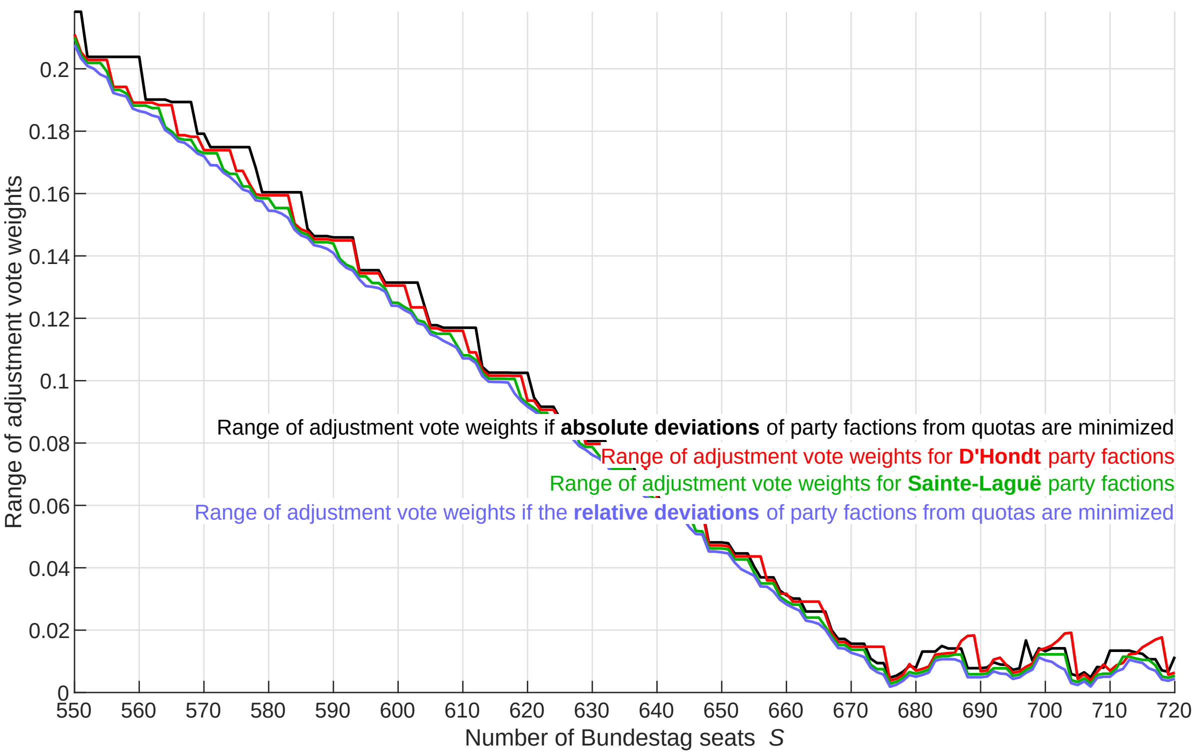


Figure 8: Range of adjustment vote weights for the Bundestag with **299 direct mandates and no SSW**.

Table 15: Pearson correlations between the curves in Figures 1 and 2.

| | SL upper | DH upper | AO upper | RO upper | SL lower | DH lower | AO lower | RO lower | SL votes | DH votes | AO votes | RO votes |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| SL upper | 1 | .44*** | .99*** | .56*** | .07 | .12 | .12 | .11 | .10 | .09 | .10 | .09 |
| DH upper | .44*** | 1 | .44*** | .83*** | .14 | −.46*** | .16* | −.30*** | −.01 | −.04 | −.00 | −.02 |
| AO upper | .99*** | .44*** | 1 | .57*** | .10 | .12 | .13 | .11 | .10 | .09 | .10 | .10 |
| RO upper | .56*** | .83*** | .57*** | 1 | .11 | −.20** | .13 | −.37*** | .18* | .17* | .19* | .17* |
| SL lower | .07 | .14 | .10 | .11 | 1 | .46*** | .99*** | .61*** | .10 | .11 | .11 | .11 |
| DH lower | .12 | −.46*** | .12 | −.20** | .46*** | 1 | .47*** | .74*** | .23** | .28*** | .24** | .24** |
| AO lower | .12 | .16* | .13 | .13 | .99*** | .47*** | 1 | .63*** | .11 | .12 | .11 | .12 |
| RO lower | .11 | −.30*** | .11 | −.37*** | .61*** | .74*** | .63*** | 1 | −.02 | .00 | −.02 | −.01 |
| SL votes | .10 | −.01 | .10 | .18* | .10 | .23** | .11 | −.02 | 1 | 1.0*** | 1.0*** | 1.0*** |
| DH votes | .09 | −.04 | .09 | .17* | .11 | .28*** | .12 | .00 | 1.0*** | 1 | 1.0*** | 1.0*** |
| AO votes | .10 | −.00 | .10 | .19* | .11 | .24** | .11 | −.02 | 1.0*** | 1.0*** | 1 | 1.0*** |
| RO votes | .09 | −.02 | .10 | .17* | .11 | .24** | .12 | −.01 | 1.0*** | 1.0*** | 1.0*** | 1 |

Table 16: Pearson correlations between the curves in Figures 3 and 4.

| | SL upper | DH upper | AO upper | RO upper | SL lower | DH lower | AO lower | RO lower | SL votes | DH votes | AO votes | RO votes |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| SL upper | 1 | .36*** | .94*** | .92*** | −.10 | .21** | .16* | −.06 | .09 | −.18* | .37*** | .11 |
| DH upper | .36*** | 1 | .34*** | .38*** | .04 | −.44*** | .13 | −.00 | .40*** | .49*** | .41*** | .40*** |
| AO upper | .94*** | .34*** | 1 | .88*** | −.02 | .21** | .13 | −.01 | .01 | −.22** | .27*** | .04 |
| RO upper | .92*** | .38*** | .88*** | 1 | −.11 | .19* | .11 | −.18* | .18* | −.16* | .43*** | .18* |
| SL lower | −.10 | .04 | −.02 | −.11 | 1 | .38*** | .91*** | .91*** | −.63*** | −.34*** | −.54*** | −.59*** |
| DH lower | .21** | −.44*** | .21** | .19* | .38*** | 1 | .43*** | .37*** | −.43*** | −.89*** | −.22** | −.41*** |
| AO lower | .16* | .13 | .13 | .11 | .91*** | .43*** | 1 | .86*** | −.57*** | −.36*** | −.41*** | −.51*** |
| RO lower | −.06 | −.00 | −.01 | −.18* | .91*** | .37*** | .86*** | 1 | −.67*** | −.33*** | −.56*** | −.61*** |
| SL votes | .09 | .40*** | .01 | .18* | −.63*** | −.43*** | −.57*** | −.67*** | 1 | .61*** | .84*** | .98*** |
| DH votes | −.18* | .49*** | −.22** | −.16* | −.34*** | −.89*** | −.36*** | −.33*** | .61*** | 1 | .43*** | .61*** |
| AO votes | .37*** | .41*** | .27*** | .43*** | −.54*** | −.22** | −.41*** | −.56*** | .84*** | .43*** | 1 | .85*** |
| RO votes | .11 | .40*** | .04 | .18* | −.59*** | −.41*** | −.51*** | −.61*** | .98*** | .61*** | .85*** | 1 |

Table 17: Pearson correlations between the curves in Figures 5 and 6.

| | SL upper | DH upper | AO upper | RO upper | SL lower | DH lower | AO lower | RO lower | SL votes | DH votes | AO votes | RO votes |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| SL upper | 1 | 1.0*** | 1.0*** | 1.0*** | −.95*** | −.89*** | −.95*** | −.95*** | 1.0*** | .99*** | 1.0*** | 1.0*** |
| DH upper | 1.0*** | 1 | 1.0*** | 1.0*** | −.94*** | −.91*** | −.95*** | −.94*** | .99*** | .99*** | .99*** | .99*** |
| AO upper | 1.0*** | 1.0*** | 1 | 1.0*** | −.95*** | −.89*** | −.95*** | −.95*** | 1.0*** | .99*** | 1.0*** | 1.0*** |
| RO upper | 1.0*** | 1.0*** | 1.0*** | 1 | −.95*** | −.89*** | −.95*** | −.95*** | 1.0*** | .99*** | 1.0*** | 1.0*** |
| SL lower | −.95*** | −.94*** | −.95*** | −.95*** | 1 | .93*** | .98*** | .95*** | −.95*** | −.95*** | −.96*** | −.95*** |
| DH lower | −.89*** | −.91*** | −.89*** | −.89*** | .93*** | 1 | .94*** | .90*** | −.90*** | −.92*** | −.90*** | −.90*** |
| AO lower | −.95*** | −.95*** | −.95*** | −.95*** | .98*** | .94*** | 1 | .96*** | −.96*** | −.96*** | −.96*** | −.96*** |
| RO lower | −.95*** | −.94*** | −.95*** | −.95*** | .95*** | .90*** | .96*** | 1 | −.96*** | −.95*** | −.96*** | −.95*** |
| SL votes | 1.0*** | .99*** | 1.0*** | 1.0*** | −.95*** | −.90*** | −.96*** | −.96*** | 1 | 1.0*** | 1.0*** | 1.0*** |
| DH votes | .99*** | .99*** | .99*** | .99*** | −.95*** | −.92*** | −.96*** | −.95*** | 1.0*** | 1 | 1.0*** | 1.0*** |
| AO votes | 1.0*** | .99*** | 1.0*** | 1.0*** | −.96*** | −.90*** | −.96*** | −.96*** | 1.0*** | 1.0*** | 1 | 1.0*** |
| RO votes | 1.0*** | .99*** | 1.0*** | 1.0*** | −.95*** | −.90*** | −.96*** | −.95*** | 1.0*** | 1.0*** | 1.0*** | 1 |

Table 18: Pearson correlations between the curves in Figures 7 and 8.

| | SL upper | DH upper | AO upper | RO upper | SL lower | DH lower | AO lower | RO lower | SL votes | DH votes | AO votes | RO votes |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| SL upper | 1 | 1.0*** | 1.0*** | 1.0*** | −.97*** | −.96*** | −.98*** | −.97*** | 1.0*** | 1.0*** | 1.0*** | 1.0*** |
| DH upper | 1.0*** | 1 | 1.0*** | 1.0*** | −.97*** | −.96*** | −.98*** | −.97*** | 1.0*** | 1.0*** | 1.0*** | 1.0*** |
| AO upper | 1.0*** | 1.0*** | 1 | 1.0*** | −.97*** | −.96*** | −.98*** | −.97*** | 1.0*** | 1.0*** | 1.0*** | 1.0*** |
| RO upper | 1.0*** | 1.0*** | 1.0*** | 1 | −.97*** | −.96*** | −.98*** | −.97*** | 1.0*** | 1.0*** | 1.0*** | 1.0*** |
| SL lower | −.97*** | −.97*** | −.97*** | −.97*** | 1 | .97*** | .98*** | .97*** | −.97*** | −.97*** | −.98*** | −.97*** |
| DH lower | −.96*** | −.96*** | −.96*** | −.96*** | .97*** | 1 | .97*** | .94*** | −.96*** | −.96*** | −.96*** | −.96*** |
| AO lower | −.98*** | −.98*** | −.98*** | −.98*** | .98*** | .97*** | 1 | .97*** | −.98*** | −.98*** | −.98*** | −.98*** |
| RO lower | −.97*** | −.97*** | −.97*** | −.97*** | .97*** | .94*** | .97*** | 1 | −.97*** | −.97*** | −.98*** | −.97*** |
| SL votes | 1.0*** | 1.0*** | 1.0*** | 1.0*** | −.97*** | −.96*** | −.98*** | −.97*** | 1 | 1.0*** | 1.0*** | 1.0*** |
| DH votes | 1.0*** | 1.0*** | 1.0*** | 1.0*** | −.97*** | −.96*** | −.98*** | −.97*** | 1.0*** | 1 | 1.0*** | 1.0*** |
| AO votes | 1.0*** | 1.0*** | 1.0*** | 1.0*** | −.98*** | −.96*** | −.98*** | −.98*** | 1.0*** | 1.0*** | 1 | 1.0*** |
| RO votes | 1.0*** | 1.0*** | 1.0*** | 1.0*** | −.97*** | −.96*** | −.98*** | −.97*** | 1.0*** | 1.0*** | 1.0*** | 1 |

*** PVAL $<$ 0.001
** 0.001 $<$ PVAL $\leq$ 0.01
* 0.01 $<$ PVAL $\leq$ 0.05

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