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by Clemens Puppe and Attila Tasnadi

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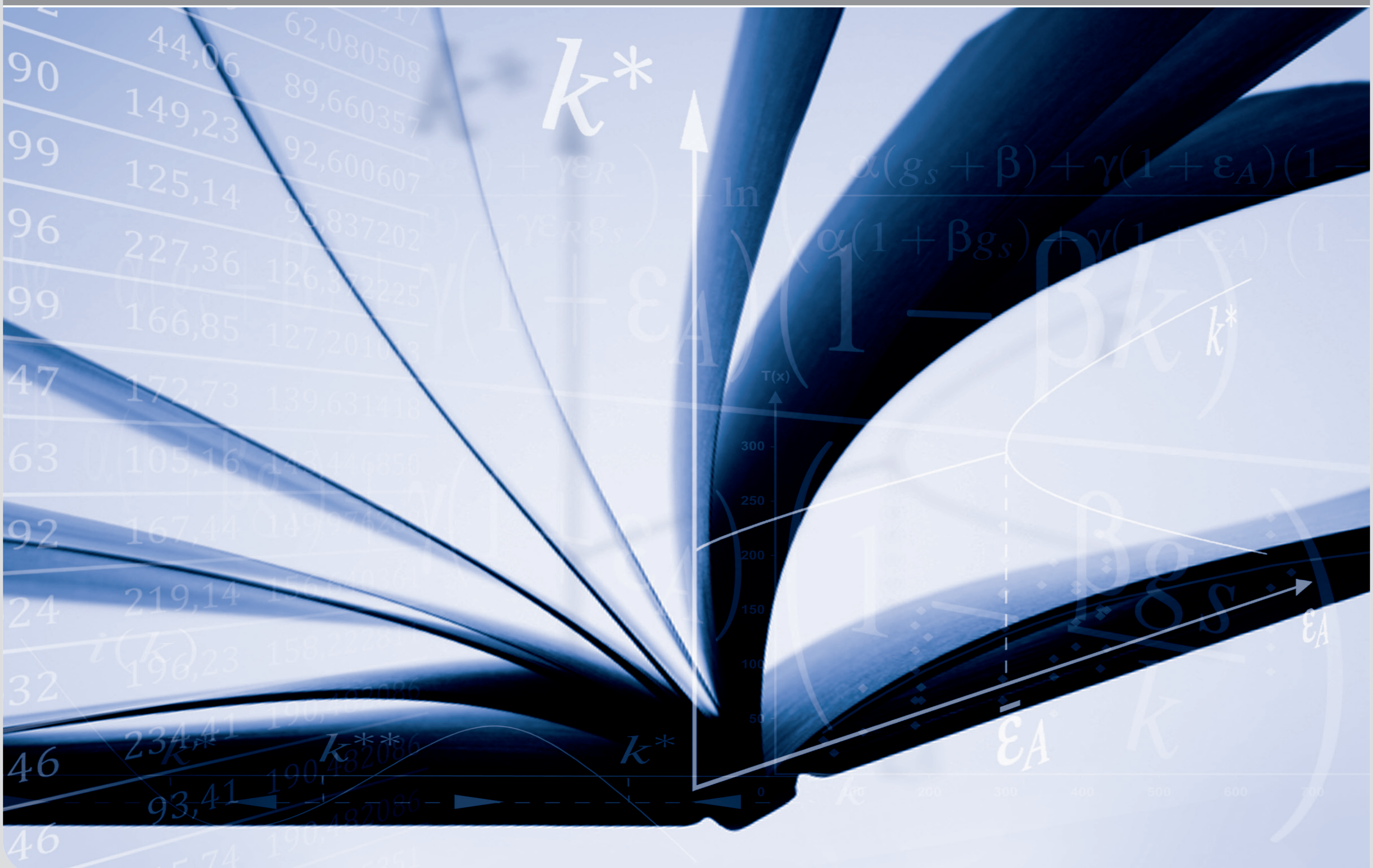

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Aggregating Incomplete Lists

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Abstract *We study the aggregation of partial orders into a complete ordering, and prove both possibility and impossibility results in this context. First, we show that the standard independence of irrelevant alternatives condition is stronger here since even dictatorial aggregation rules may fail to satisfy it. On the other hand, domain restrictions enable non-dictatorial aggregation rules satisfying a number of attractive properties. In particular, we show that anonymous aggregation satisfying a weak form of independence of irrelevant alternatives is possible on a large class of 'extended' Condorcet domains.*

Keywords: Multi-criteria decision making, aggregation of partial orders and incomplete lists, Arrow's theorem.
***JEL* Classification Number:** D71.

1 Introduction

We study the aggregation of (strict) partial orders into a collective ranking. Applications of this general problem abound. Indeed, many problems require the aggregation of rankings of objects based on inputs from multiple sources like in automated decision making, machine learning (Volkovs and Zemel, 2014), database middleware (Masthoff, 2004), search engines and spam filters (Dwork et al., 2002), or in the determination of the results in sport competitions (Csató, 2023). The problem also arises in coding theory since the alternatives can be regarded as letters and the rankings as strings (Bortolussi et al., 2012). The issue of aggregating partial rankings also emerges in the link analysis in networks and web search algorithms (Borodin et al., 2005). Further applications dealing with the aggregation of incomplete preferences range from student paper competitions (Hochbaum and Moreno-Centeno, 2021), the ranking of cities as destination for tourists (Dopazo and Martìnez-Céspedes, 2015) to the ranking of teams in sports competition (Ausloos, 2024). For an application to the ranking of individual tennis players of different decades, see Bozoki et al. (2016) and Temesi et al. (2024); the issue of ranking violin virtuosi based on views of their performances on Youtube has been addressed by Puppe and Tasnádi (2025).

In social choice theory, the problem of aggregating incomplete orders has been addressed by a large number of scholars, see e.g. Pini et al. (2009) for a comprehensive treatment of three central results under incomplete preferences: Arrow's theorem (Arrow 1951), the Gibbard-Satterthwaite theorem (Gibbard 1973, Satterthwaite 1975) and the Muller-Satterthwaite theorem (Muller and Satterthwaite, 1977). In particular, these authors prove a variant of Arrow's impossibility theorem if both individual and collective rankings are incomplete. The present paper departs from their approach in two ways. First, we insist in the completeness of the collective ranking. This is motivated by the fact that ultimately a choice has to be made and the possible incompleteness is thus necessarily resolved, at least at the top of the collective ranking. If one then, hypothetically, considers every possible submenu as the feasible set from which choices are made, all incompleteness is eventually resolved.

Secondly, we allow for domains of profiles of individual rankings that are not of the form of a Cartesian product. Indeed, in many applications one may want to allow for systematic correlations between voters and their possible (incomplete) rankings. For instance, it could be known that some types of voters have complete rankings, while others exhibit systematic lack of completeness. For instance, it seems natural to allow for the possibility that different voters have different expertise over different subsets of candidates, and are able to rank only those candidates they are familiar with.

We introduce different variants of Arrow's independence of irrelevant alternatives (IIA) condition, based on which we prove various impossibility results and a possibility result. Our main insight is that in our context even dictatorial social welfare functions violate IIA if we allow for some extent of incompleteness of the individual rankings. Thus, Arrow's conclusion is strengthened in that *no* social welfare function can satisfy IIA and the weak Pareto principle if individual rankings exhibit some degree of incompleteness.

It has often been informally argued in the literature that enlarging only the domain of a SWF cannot escape the impossibility, but we are not aware of a formal proof of this statement. We provide such a formal proof and demonstrate that in fact an additional condition of 'information monotonicity' is needed to give a rigorous account of this 'folk' intuition.

Finally, we also provide a possibility result that escapes the Arrow impossibility. Specifically, we identify a large class of 'extended' Condorcet domains on which pairwise majority voting always a complete (and transitive) collective ranking.

2 Basic Definitions and Conditions

Let $A = \{a_1, \ldots, a_m\}$ be the set of alternatives and $N = \{1, \ldots, n\}$ the set of voters with $n \geq 2$. By $\mathcal{I}$ we denote the set of all strict partial orders (asymmetric and transitive binary relations) on A, i.e. the set of possibly incomplete individual (strict) preferences, and by $\mathcal{P}$ the set of all linear orders (asymmetric, transitive and complete[1] binary relations) on A, i.e. the set of complete individual strict preferences.

In this paper, we are interested in domains of profiles of strict partial orders $\mathfrak{D}$ satisfying $\mathcal{P}^n \subseteq \mathfrak{D} \subseteq \mathcal{I}^n$, and we will assume this from now on. An important example is the domain of all profiles of 'lists.'

Definition 1. A *list (on A)* is a strict partial order $\succ \subseteq A \times A$ such that $\succ$ is complete on some subset $B \subseteq A$ with $\#B \geq 2$.

We denote by $\mathcal{L}$ the set of all (possibly incomplete) lists on A, and by $\mathcal{L}^n$ the domain of all profiles of individual lists on A.

A social welfare function defined on a subdomain $\mathfrak{D} \subseteq \mathcal{I}^n$ assigns to each profile of strict partial orders $\Pi = (\succ_1, \ldots, \succ_n) \in \mathfrak{D}$ a linear order, i.e. a complete social ranking of all alternatives in A. Formally, we have the following definition.

Definition 2. A mapping $F : \mathfrak{D} \to \mathcal{P}$ is called a *social welfare function*, henceforth, SWF on the domain $\mathfrak{D}$.

We turn to the appropriate generalizations of the well-known notions of Pareto property, independence of irrelevant alternatives and dictatorship in our present context.

The first is the weak Pareto property stating that if all voter rank a above b, then so must the social ranking.

Definition 3. A SWF $F : \mathfrak{D} \to \mathcal{P}$ satisfies the *weak Pareto property* (WP) if, for all profiles $\Pi = (\succ_1, \ldots, \succ_n) \in \mathfrak{D}$ and $a, b \in A$ we have

$$(\forall i \in N : a \succ_i b) \implies a \succ b,$$

where $\succ = F(\succ_1, \ldots, \succ_n)$.

In our context, the natural notion of dictatorship is as follows.

Definition 4. A SWF $F : \mathfrak{D} \to \mathcal{P}$ is *dictatorial* if there exists a voter h such that, for all $(\succ_1, \ldots, \succ_n) \in \mathfrak{D}$ and all $a, b \in A$ we have

$$a \succ_h b \implies a \succ b,$$

where $\succ = F(\succ_1, \ldots, \succ_n)$.

[1] An order $\succ \subseteq A \times A$ is complete if, for all distinct $x, y \in A$, either $x \succ y$ or $y \succ x$; sometimes this condition is also called 'connectedness.'

Observe that a dictator thus imposes the ordering of those alternatives on which she has an opinion. Clearly, there can be at most one such voter. Also note that in our context it is not possible to define a dictator as a voter who imposes *exactly* her (incomplete) preference as the social ranking since we assume the social ranking always to be complete.

Let us consider next the extension of the well-known IIA condition, which requires that, for any pair of distinct alternatives, if in two profiles these two alternatives are ranked in the same way voter by voter, then the SWF must rank these two alternatives in both profiles in the same way. The following strong version of this requirement formally resembles the well-known formulation.

Definition 5. The SWF $F : \mathfrak{D} \to \mathcal{P}$ satisfies *strong independence of irrelevant alternatives (SIIA)* if, for all distinct $a, b \in A$ and all $\Pi = (\succ_1, \dots, \succ_n), \Pi' = (\succ'_1, \dots, \succ'_n) \in \mathfrak{D}$ we have

$$\big(\forall i \in N : a \succ_i b \Leftrightarrow a \succ'_i b\big) \implies \big(a \succ b \Leftrightarrow a \succ' b\big)\,, \tag{2.1}$$

where $\succ = F(\succ_1, \dots, \succ_n)$ and $\succ' = F(\succ'_1, \dots, \succ'_n)$.

Note that the condition does not distinguish, for a given pair $a, b \in A$, between voters who strictly prefer b to a from voters who do not rank a versus b as long as the set of voters who strictly prefer a to b remains the same. Therefore, the following weaker version seems in fact more appropriate in our context.

Definition 6. The SWF $F : \mathfrak{D} \to \mathcal{P}$ satisfies *independence of irrelevant alternatives (IIA)* if, for all distinct $a, b \in A$, and all profiles $\Pi = (\succ_1, \dots, \succ_n), \Pi' = (\succ'_1, \dots, \succ'_n) \in \mathfrak{D}$ we have

$$\big(\forall i \in N : a \succ_i b \Leftrightarrow a \succ'_i b \text{ and } b \succ_i a \Leftrightarrow b \succ'_i a\big) \implies \big(a \succ b \Leftrightarrow a \succ' b\big)\,, \tag{2.2}$$

where $\succ = F(\succ_1, \dots, \succ_n)$ and $\succ' = F(\succ'_1, \dots, \succ'_n)$.

Clearly, if F satisfies SIIA then it also satisfies IIA since the antecedent of (2.2) implies the antecedent of (2.1); the converse does not hold.

3 Which Social Choice Rules Satisfy SIIA?

Arrow's famous theorem states that on the domain $\mathcal{P}^n$ the SWFs that satisfy WP and SIIA are exactly the dictatorial ones. Here, we show that on larger domains in general *no* SWF can satisfy WP and SIIA. First, let us note the following simple generalization of Arrow's theorem.

Theorem 1. *Suppose that $m \geq 3$ and $\mathcal{P}^n \subseteq \mathfrak{D}$. Every SWF $F : \mathfrak{D} \to \mathcal{P}$ satisfying the weak Pareto property (WP) and strong independence of irrelevant alternatives (SIIA) is dictatorial.*

Proof. Suppose that F satisfies WP and SIIA. First, consider the restriction of $F|_{\mathcal{P}^n}$ of F to the subdomain of all complete preference orderings. As is easily verified $F|_{\mathcal{P}^n} : \mathcal{P}^n \to \mathcal{P}$ satisfies SIIA and WP on the subdomain of all complete preference. Hence, by Arrow's original theorem $F|_{\mathcal{P}^n}$ is dictatorial. Denote the dictator on the subdomain $\mathcal{P}^n$ by $h \in N$.

Consider now any profile $(\succ_1, \dots, \succ_n) \in \mathfrak{D}$ and any pair $a, b \in A$, and assume that $a \succ_h b$. For all $i \in N$ with $a \succ_i b$ or $b \succ_i a$ consider any linear extension $\succ'_i$ of $\succ_i$. For all $i \in N$ such that a and b are not ranked by $\succ_i$ choose a linear extension $\succ'_i$ of $\succ_i$ that ranks b

above a (such extension exists by Szpilrajn's well-known theorem). By construction, we have $\{i : a \succ_i b\} = \{i : a \succ'_i b\}$, hence by SIIA $a \succ b \Leftrightarrow a \succ' b$ where $\succ = F(\succ_1, \ldots, \succ_n)$ and $\succ' = F(\succ'_1, \ldots, \succ'_n)$. Since $\succ'_h$ extends $\succ_h$, and since the profile $(\succ'_1, \ldots, \succ'_n)$ belongs to $\mathcal{P}^n$, we have $a \succ' b$. Therefore also $a \succ b$. This shows that F is dictatorial (with dictator h). □

In contrast to the standard framework of complete orderings, dictatorial SWFs do not automatically satisfy SIIA in our framework. In fact, on the domain $\mathcal{I}$ of all partial orders there does not exist *any* SWF satisfying WP and SIIA.

Theorem 2. *Suppose that $m \geq 3$. No SWF $F : \mathcal{I}^n \to \mathcal{P}$ can simultaneously satisfy the weak Pareto property (WP) and strong independence of irrelevant alternatives (SIIA).*

Proof. By contradiction, suppose that $F : \mathcal{I} \to \mathcal{P}$ satisfies WP and SIIA. By Theorem 1, F is dictatorial; let h be the dictator. Consider any profile $\Pi = (\succ_1, \ldots, \succ_n)$ such that $\succ_h$ is the empty partial order that refrains from any binary comparisons. Since F maps into the set of (complete) linear orders, we have $F(\Pi) =: \succ \in \mathcal{P}$. Let $a, b, c \in A$ such that $a \succ b \succ c$. Denote by $>^{ca}$ the partial order that only declares c superior to a while refraining from any other binary comparison. Consider the profile Π' that coincides with Π except that $\succ_h = \emptyset$ is replaced by $\succ'_h = >^{ca}$, and let $\succ' := F(\Pi')$. We have $a \succ b$ and $\{i : a \succ_i b\} = \{i : a \succ'_i b\}$, hence by SIIA, $a \succ' b$. Similarly, we have $b \succ c$ and $\{i : b \succ_i c\} = \{i : b \succ'_i c\}$, hence by SIIA, $b \succ' c$, hence by transitivity $a \succ' c$. But this contradicts the assumption that F is dictatorial with dictator h. □

One can generalize the conclusion of Theorem 2 to all sets of profiles that are 'minimally rich' in the following sense. As in the proof of Theorem 2 denote by $>^{xy} := \{(x, y)\}$, i.e. the partial order that only compares the pair $x, y \in A$ and declares x superior to y.

Definition 7. Say that a set of profiles $\mathfrak{D}$ is *minimally rich* if, for all $i \in N$ and all distinct pairs $x, y \in A$, there exists a profile $\Pi \in \mathfrak{D}$ that assigns the order $>^{xy}$ to voter i.

Note that the domains $\mathcal{L}^n$ and $\mathcal{I}^n$ are both minimally rich, while $\mathcal{P}^n$ is not. The following lemma is central to our analysis.

Lemma 3.1. *Let $m \geq 3$, and suppose that $\mathfrak{D}$ is minimally rich and $F : \mathfrak{D} \to \mathcal{P}$ is dictatorial. Then, F violates (weak) IIA.*

Proof. By contradiction, suppose that $F : \mathfrak{D} \to \mathcal{P}$ is dictatorial with dictator h, and that F satisfies IIA. Let $a, b, c \in A$ and fix, for any pair of distinct $x, y \in A$, a profile Π_h^{xy} as required in Definition 7, i.e. a profile that assigns the order $>^{xy}$ to the dictator h. For all distinct $x, y \in \{a, b, c\}$, we denote $F(\Pi_h^{xy}) := \succ^{xy}$ (not to be confused with $>^{xy}$). Observe that all orders $\succ^{xy}$, $x, y \in \{a, b, c\}$, are complete. In particular, $\succ^{ac}$ must compare both pairs a, b and b, c. We derive a contradiction in all possible cases.

Case 1. Suppose that $a \succ^{ac} b$ and $b \succ^{ac} c$. By IIA, we must then also have $a \succ^{ca} b$ and $b \succ^{ca} c$, hence by transitivity $a \succ^{ca} c$ which contradicts the assumption that F is dictatorial with dictator h.

Case 2. Suppose that $a \succ^{ac} b$ and $c \succ^{ac} b$. By IIA we obtain $a \succ^{bc} b$ from $a \succ^{ac} b$ (because neither $>^{ac}$ nor $>^{bc}$ compare the pair a, b); moreover, $b \succ^{bc} c$ because h is a dictator, hence by transitivity $a \succ^{bc} c$. Similarly, by SIIA we obtain $c \succ^{ba} b$ from $c \succ^{ac} b$ (because neither $>^{ac}$ nor $>^{ba}$ compare the pair b, c); moreover, $b \succ^{ba} a$ because h is a dictator, hence by transitivity $c \succ^{ba} a$. Summarizing, we have both $a \succ^{bc} c$ and $c \succ^{ba} a$, but this contradicts IIA.

Case 3. Suppose that $b \succ^{ac} a$ and $b \succ^{ac} c$. By IIA we obtain $b \succ^{cb} a$ from $b \succ^{ac} a$ (because neither $>^{ac}$ nor $>^{cb}$ compare the pair a, b); moreover, $c \succ^{cb} b$ because h is a dictator, hence by transitivity $c \succ^{cb} a$. Similarly, by IIA we obtain $b \succ^{ab} c$ from $b \succ^{ac} c$ (because neither $>^{ac}$ nor $>^{ab}$ compare the pair b, c); moreover, $a \succ^{ab} b$ because h is a dictator, hence by transitivity $a \succ^{ab} c$. Summarizing, we have both $c \succ^{cb} a$ and $a \succ^{ab} c$, but this contradicts IIA.

Case 4. Suppose that $b \succ^{ac} a$ and $c \succ^{ac} b$. By transitivity, we have $c \succ^{ac} a$ which directly contradicts the assumption that h is a dictator.

In all possible cases we thus obtain a contradiction. □

Combining Theorem 1 and Lemma 3.1 we obtain the following result.

Theorem 3. *Suppose that $m \geq 3$, and that $\mathfrak{D}$ is minimally rich. No SWF $F : \mathfrak{D} \to \mathcal{P}$ can simultaneously satisfy the weak Pareto property (WP) and strong independence of irrelevant alternatives (SIIA).*

Remark 1. While the conditions WP and SIIA cannot be satisfied simultaneously, they can clearly be satisfied in isolation. For instance, the SWF that always yields a single fixed linear order clearly satisfies SIIA (but not WP). On the other hand, WP can be satisfied as follows. For any partial order $\succ$ let $g(\succ) \in \mathcal{P}$ be a linear extension of $\succ$ (such extension exists due to Szpilrajn's extension theorem). Then, every SWF that assigns to every profile $(\succ_1, \ldots, \succ_n) \in \mathfrak{D}$ the order $g(\cap_{i=1}^{n} \succ_i)$ satisfies WP (but by Theorem 3, no such SWF can satisfy SIIA if $\mathfrak{D}$ is minimally rich).

4 Which Social Choice Rules Satisfy IIA?

Do we get similar results if we weaken the strong SIIA condition to the more plausible (weak) IIA condition? Not quite, as we shall show now. First, consider the following example.

Example: A Non-Dictatorial SWF satisfying WP and IIA

Let $n = 2$ and $A = \{a, b, c\}$. We define a SWF F through Tables 1-3 as follows.

Note that although voter 2 ranks c on top in the right profile and does not rank c in the left profile, c drops in the collective order from the left to the right profile in Table 1. In particular, neither voter 1 nor voter 2 is a dictator.

Next, we define F if voter 2 inserts c at another rank (starting from the order $>^{ba}$ in the left profile in Table 1):

For the three profiles in which voter 1 inserts a, the F maintains the collective ranking cba from the left profile in Table 1:

$\succ_1$	$\succ_2$	$F(\succ_1, \succ_2)$
b	b	c
c	a	b
		a

$\succ_1$	$\succ_2$	$F(\succ_1, \succ_2)$
b	c	b
c	b	c
	a	a

Table 1: Voter 2 inserts c at the top

$\succ_1$	$\succ_2$	$F(\succ_1, \succ_2)$
b	b	b
c	a	c
	c	a

$\succ_1$	$\succ_2$	$F(\succ_1, \succ_2)$
b	b	b
c	c	c
	a	a

Table 2: Voter 2 inserts c at the bottom and in the middle

For any other profile in $\mathcal{P}^n$ let voter 1 be the dictator. Finally, let $\mathfrak{D}$ be the union of $\mathcal{P}^n$ and the seven profiles in Tables 1-3. It can be verified that F satisfies WP and IIA on $\mathfrak{D}$. Also observe that while the domain is not a Cartesian product, it does satisfy $\mathcal{P}^n \subseteq \mathfrak{D} \subseteq \mathcal{L}^n$, i.e. all individual orders are lists.

Although non-dictatorial, the SWF F defined in the example exhibits an implausible violation of monotonicity. Indeed, as already noted, voter 2's putting the unranked alternative c on top leads to a drop of that alternative in the collective ranking. The following 'information monotonicity' condition forbids this; it states that additional individual preference information confirming the social ranking will reinforce that ranking.

Definition 8. A SWF $F : \mathfrak{D} \to \mathcal{P}$ is *information monotonic (IM)* if, for all distinct $a, b \in A$, all $i \in N$, and all $\Pi = (\succ_1, \ldots, \succ_n), \Pi' = (\succ'_1, \ldots, \succ'_n) \in \mathfrak{D}$ such that $\succ_i \subseteq \succ'_i$ and $\succ_j = \succ'_j$ for all $j \neq i$ we have

$$\big(a \succ b \text{ and } a \succ'_i b\big) \implies a \succ' b,$$

where $\succ = F(\succ_1, \ldots, \succ_n)$ and $\succ' = F(\succ'_1, \ldots, \succ'_n)$.

Note that this condition is even weaker than the usual requirements of positive responsiveness since the IM has bite only in cases in which a single voter's preference is *extended* (while all other voters' preferences are held fixed). Moreover, it is implied by SIIA.

Proposition 1. *Every SWF $F : \mathfrak{D} \to \mathcal{P}$ that satisfies SIIA also satisfies IM.*

Proof. Suppose that the profiles $(\succ_1, \ldots, \succ_n)$ and $(\succ'_1, \ldots, \succ'_n)$ in $\mathfrak{D}$ are such that $\succ_i \subseteq \succ'_i$ and $\succ_j = \succ'_j$ for all $j \neq i$, and suppose that $a \succ'_i b$ and $a \succ b$ where $\succ = F(\succ_1, \ldots, \succ_n)$. By contradiction, assume that $b \succ' a$ where $\succ' = F(\succ'_1, \ldots, \succ'_n)$. We have $\{j : b \succ_j a\} = \{j : b \succ'_j a\}$, hence by SIIA, $b \succ a \Leftrightarrow b \succ' a$ in contradiction to the assumption $a \succ b$. $\square$

$\succ_1$	$\succ_2$	$F(\succ_1, \succ_2)$
b	b	c
c	a	b
a		a

$\succ_1$	$\succ_2$	$F(\succ_1, \succ_2)$
b	b	c
a	a	b
c		a

$\succ_1$	$\succ_2$	$F(\succ_1, \succ_2)$
a	b	c
b	a	b
c		a

Table 3: Voter 1 inserts a

Definition 9. Say that a set of profiles $\mathfrak{D}$ is *pairwise extendable* if, for all profiles $\Pi = (\succ_1, \ldots, \succ_n) \in \mathfrak{D}$ and all pairs a, b such that $\succ_i$ does not rank a versus b, there exists an extension $\succ_i'$ of $\succ_i \cup \{(a, b)\}$ such that $\Pi' \in \mathfrak{D}$, where Π' is the profile that results from Π by replacing $\succ_i$ by $\succ_i'$ (keeping all other orders fixed).

Evidently, all domains $\mathcal{I}^n, \mathcal{L}^n, \mathcal{P}^n$ are pairwise extendable, as is the domain defined in the example above.

We are now ready to state the two main results of this section.

Theorem 4. *Suppose that $m \geq 3$, $\mathcal{P}^n \subseteq \mathfrak{D}$ and that $\mathfrak{D}$ is pairwise extendable. Every SWF $F : \mathfrak{D} \to \mathcal{P}$ satisfying the weak Pareto property (WP), independence of irrelevant alternatives (IIA) and information monotonicity (IM) is dictatorial.*

Proof. First, consider the restriction of $F|_{\mathcal{P}^n}$ of F to the subdomain of all complete preference orderings. As is easily verified $F|_{\mathcal{P}^n} : \mathcal{P}^n \to \mathcal{P}$ satisfies SIIA and WP on the subdomain of all complete preference. Hence, by Arrow's original theorem $F|_{\mathcal{P}^n}$ is dictatorial. Denote the dictator on the subdomain $\mathcal{P}^n$ by $h \in N$. We will prove that h is a dictator on $\mathfrak{D}$ by contradiction.

Thus, suppose that h is not a dictator on $\mathfrak{D}$. Then there exist a profile $\Pi = (\succ_1, \ldots, \succ_n) \in \mathfrak{D}$ and two distinct alternatives $a, b \in A$ such $a \succ_h b$ and $b \succ a$ where $\succ = F(\succ_1, \ldots, \succ_n)$. Evidently, Π cannot be an element of $\mathcal{P}^n$. First, by pairwise extendability there exists a profile $\Pi' = (\succ_1', \ldots, \succ_n') \in \mathfrak{D}$ such that, for all $i \in N$, (i) $\succ_i' = \succ_i$ if either $a \succ_i b$ or $b \succ_i a$ and (ii) $\succ_i' \supseteq (\succ_i \cup (b, a))$ if $a \not\succ_i b$ and $b \not\succ_i a$. By IM we have $b \succ' a$ where $\succ' = F(\succ_1', \ldots, \succ_n')$. Next, extend Π' to a profile $\Pi'' \in \mathcal{P}^n$ such that, for all $i \in N$, $\succ_i'' \supseteq \succ_i'$ (this is possible due to Szpilrajn's extension theorem). By assumption, we have $\Pi'' \subseteq \mathfrak{D}$, and by IIA we have $b \succ'' a$ where $\succ'' = F(\succ_1'', \ldots, \succ_n'')$. But this contradicts the fact that individual h is a dictator on the subdomain $\mathcal{P}^n$. □

Finally, combining Theorem 4 and Lemma 3.1, we obtain the following result.

Theorem 5. *Suppose that $m \geq 3$, and that $\mathfrak{D}$ is minimally rich and pairwise extendable. No SWF $F : \mathfrak{D} \to \mathcal{P}$ can simultaneously satisfy the weak Pareto property (WP), independence of irrelevant alternatives (IIA) and information monotonicity (IM).*

Noting that the domain of all profiles of lists satisfies both the minimal richness property and pairwise extendability, we obtain the following corollary.

Corollary 1. *Suppose that $m \geq 3$. No SWF $F : \mathcal{L}^n \to \mathcal{P}$ can simultaneously satisfy the weak Pareto property (WP), independence of irrelevant alternatives (IIA) and information monotonicity (IM).*

5 A Possibility Result on Extended Condorcet Domains

If we give up the assumption that $\mathfrak{D}$ contains all of $\mathcal{P}^n$, possibility results arise. In the following, we concentrate on (a version of) pairwise majority rule as aggregation method and show that it provides an anonymous aggregation method on a large class of 'extended Condorcet domains.' Throughout this section, we assume that the number of voters n is odd.

Definition 10. A domain $\mathfrak{D} \subseteq \mathcal{P}^n$ of linear orders is called a *Condorcet domain (of profiles)* if, for all profiles $\Pi = (\succ_1, \ldots, \succ_n) \in \mathfrak{D}$, the pairwise majority relation $\succ_\Pi^{maj}$ is transitive, where

$$x \succ_\Pi^{maj} y \ :\Longleftrightarrow\ \#\{i : x \succ_i y\} > \#\{i : y \succ_i x\}.$$

(Then, $\succ_\Pi^{maj}$ is indeed a linear order due to the assumption that n is odd.) A domain $\mathfrak{D} \subseteq \mathcal{I}^n$ is called an *extended Condorcet domain (of profiles)* if there exists an *extension function* $g : \mathcal{I} \to \mathcal{P}$ with $\succ_i \subseteq g(\succ_i)$ for all $\succ_i \in \mathcal{I}$, such that the pairwise majority relation $\succ_{g(\Pi)}^{maj}$ is transitive for all profiles $\Pi = (\succ_1, \ldots, \succ_n) \in \mathfrak{D}$, where $g(\Pi) := (g(\succ_1), \ldots, g(\succ_n))$. Any such function g is called *majority admissible*.

Well-known examples of Condorcet domains are the domains of all single-peaked profiles, the domain of all single-crossing profiles, and the domain of all group-separable profiles. Thus, an example of an extended Condorcet domain of profiles is the domain of all profiles of strict partial orders that can be extended to a single-peaked profile of linear orders. Of course, the majority relation on such a profile will depend on the chosen extension function g. For instance, the profile that assigns the empty partial order to each voter can be extended in many different ways to a single-peaked profile of linear orders.

For each extended Condorcet domain $\mathfrak{D}$ and each majority admissible extension function $g : \mathcal{I} \to \mathcal{P}$, denote by F_g the SWF defined by

$$F_g(\succ_1, \ldots, \succ_n) \ := \succ_{g(\Pi)}^{maj}, \tag{5.3}$$

Evidently, every SWF $F_g : \mathfrak{D} \to \mathcal{P}$ of this form is anonymous, hence in particular non-dictatorial, and satisfies the weak Pareto property WP. However, since F_g depends in general on the extension function g it does not satisfy the IIA condition introduced above. It does, however, satisfy the following *restricted independence of irrelevant alternatives (RIIA)* condition.

Definition 11. The SWF $F : \mathfrak{D} \to \mathcal{P}$ satisfies *restricted independence of irrelevant alternatives (IIA)* if, for all distinct $a, b \in A$, and all profiles $\Pi = (\succ_1, \ldots, \succ_n), \Pi' = (\succ_1', \ldots, \succ_n') \in \mathfrak{D}$ such that, for all i, either $a \succ_i b$ or $b \succ_i a$, we have

$$\big(\forall i \in N : a \succ_i b \Leftrightarrow a \succ_i' b \text{ and } b \succ_i a \Leftrightarrow b \succ_i' a\big) \implies \big(a \succ b \Leftrightarrow a \succ' b\big), \tag{5.4}$$

where $\succ = F(\succ_1, \ldots, \succ_n)$ and $\succ' = F(\succ_1', \ldots, \succ_n')$.

Thus, RIIA weakens IIA by requiring the implication (5.4) for $a, b \in A$ only for those profiles at which no voter deems a and b as incomparable; in particular on the domain $\mathcal{P}^n$ all three conditions SIIA, IIA and RIIA are equivalent. An inspection of the proof of Theorem 4 shows that, if $m \geq 3$ and $\mathfrak{D}$ is pairwise extendable, then every SWF satisfying WP, RIIA and IM is dictatorial. Thus, Theorems 4 and 5 remain valid if the IIA condition is replaced by the weaker RIIA condition.

Theorem 6. *Let $\mathfrak{D}$ be an extended Condorcet domain of profiles, and let $g : \mathfrak{D} \to \mathcal{P}^n$ be any majority admissible extension function. Then, the SWF $F_g : \mathfrak{D} \to \mathcal{P}$ defined by (5.3) is anonymous and satisfies WP, RIIA and IM.*

Proof. By definition, F_g is anonymous and satisfies WP. (To see that it satisfies WP simply note that if $a \succ_i b$ for all i, then $ag(\succ_i)b$ for all i since $g(\succ_i)$ extends $\succ_i$.)

The SWF F_g also satisfies RIIA, as follows. For all profiles $\Pi = (\succ_1, \ldots, \succ_n)$ and all $a, b \in A$ such that $\{i : a \succ_i b\} \cup \{i : b \succ_i a\} = N$, we have

$$\{i : a \succ_i b\} \;=\; \{i : ag(\succ_i)b\} \quad \text{and} \; \{i : b \succ_i a\} \;=\; \{i : bg(\succ_i)a\},$$

since $g(\succ_i)$ extends $\succ_i$. Hence, if Π and Π' are as required in the definition of RIIA, we obtain

$$a \succ^{maj}_{g(\Pi)} b \iff a \succ^{maj}_{g(\Pi')} b,$$

i.e. condition RIIA.

Finally, let Π and Π' be as required in the definition of IM. Assume that $a \succ^{maj}_{g(\Pi)} b$, and that $a \succ'_i b$. Since $g(\succ'_i)$ extends $\succ'_i$, we have $ag(\succ'_i)b$, hence

$$\{j : ag(\succ_j)b\} \;\subseteq\; \{j : ag(\succ'_j)b\}.$$

Together with the assumption $a \succ^{maj}_{g(\Pi)} b$ this implies $a \succ^{maj}_{g(\Pi')} b$, i.e. condition IM. □

6 Conclusion

In this paper, we have investigated the Arrow problem of aggregating incomplete individual rankings to a complete collective ranking under various domain restrictions. We have shown that, if the domain contains all profiles of complete individual rankings, under mild additional conditions not even dictatorial rules can satisfy the weak Pareto principle and plausible variants of the IIA condition. On the other hand, possibility results emerge by suitably restricting the domain of a social welfare function. Specifically, we have introduced the notion of 'extended' Condorcet domains of profiles that guarantee transitivity (and completeness) of the collective ranking. It appears to be a worthwhile task for future research to investigate if there are other interesting ways to achieve possibility results for the problem of aggregating partial orders into a complete collective ranking.

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