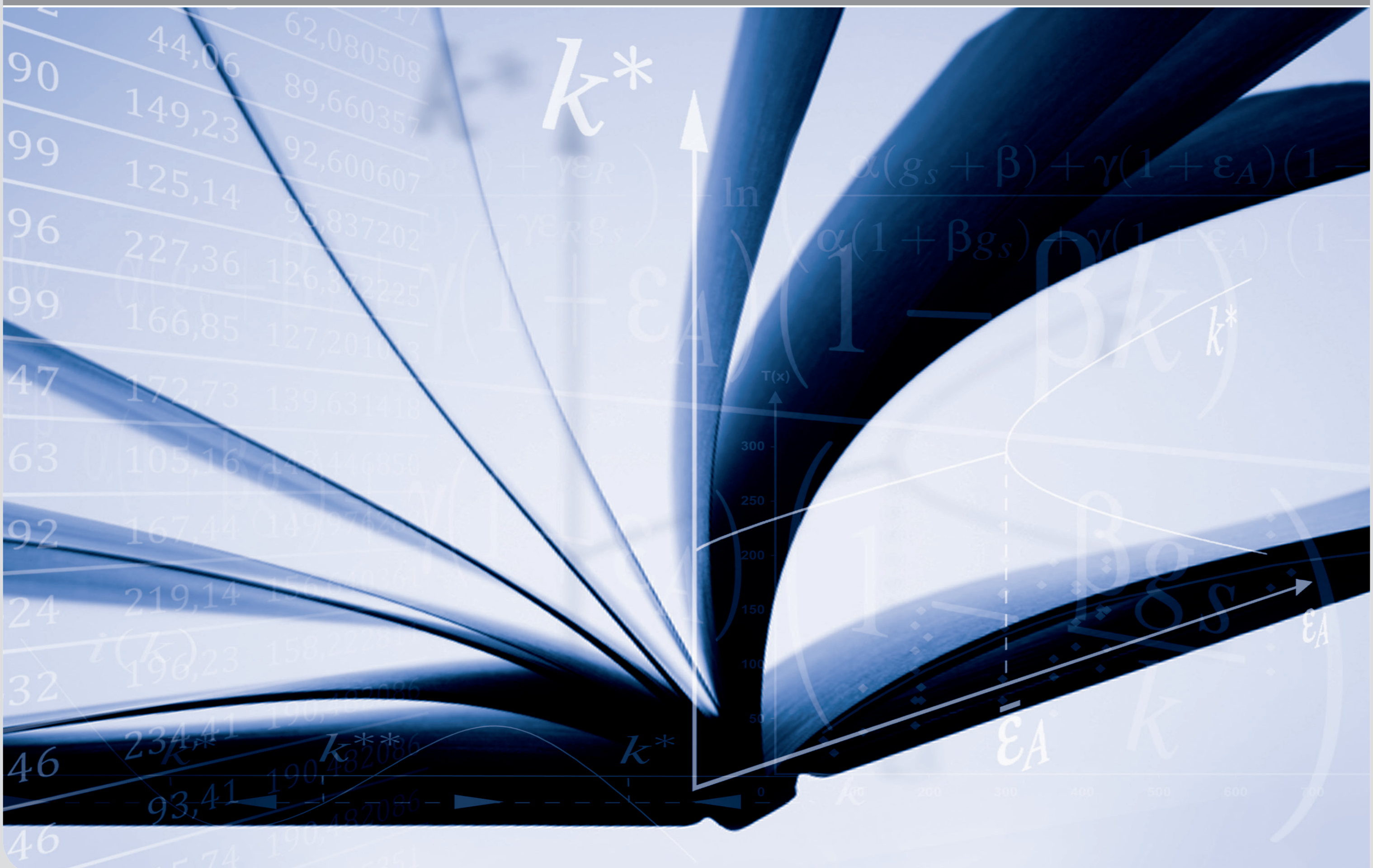
KIT
Karlsruher Institut für Technologie

Arrovian independence and the aggregation of choice functions

by Claudio Kretz and Clemens Puppe

No. 173 | JULY 2025

WORKING PAPER SERIES IN ECONOMICS



KIT – The Research University in the Helmholtz Association

econpapers.wiwi.kit.edu

Imprint

Karlsruher Institute of Technology (KIT)
Department of Economics and Management
Institute of Economics (ECON)

Kaiserstraße 12
76131 Karlsruhe

KIT – The Research University in the Helmholtz Association

Working Paper Series in Economics
No. 173, July 2025

ISSN 2190-9806

econpapers.wiwi.kit.edu

Arrovian Independence and the Aggregation of Choice Functions

Claudio Kretz[*] & Clemens Puppe[†]

July, 2025

We reappraise the Arrow problem by studying the aggregation of choice functions. We do so in the general framework of judgment aggregation, in which choice functions are naturally representable by specifying, for each menu A and each alternative x in A, whether x is choosable from A, or not. Our framework suggests a natural strengthening of Arrow's independence condition positing that the collective choosability of an alternative from a menu depends on the individual views on that issue, and that issue alone. Our analysis reveals that Arrovian impossibility results crucially hinge on what internal consistency requirements we impose on choice functions. While the aggregation of 'binary' choice functions, i.e. those satisfying both contraction (α) and expansion (γ) consistency, is necessarily dictatorial, possibilities in the form of oligarchic rules emerge for path-independent choice functions, that is, when the expansion property γ is replaced by the so-called Aizerman condition. Remarkably, the Arrovian aggregation of choice functions is shown to be almost dictatorial already under property γ alone. When giving up expansion consistency, specific quota rules become possible.

Keywords: choice function – rationalizability – aggregation theory – independence – Arrow's Theorem

JEL Classification: D01, D71

---

[*]Department of Economics and Management, Karlsruhe Institute of Technology, 76131 Karlsruhe, Germany, claudio.kretz@kit.edu.

[†]Department of Economics and Management, Karlsruhe Institute of Technology, 76131 Karlsruhe, Germany, clemens.puppe@kit.edu and
Corvinus Institute for Advanced Studies (CIAS), Corvinus University of Budapest, 1093 Budapest, Hungary.
We would like to thank the audiences of the 17th Meeting of the Society for Social Choice and Welfare held in Paris in 2024, the CMSS seminar at the University of Auckland in March 2025, and the conference in honor of Hervé Moulin's 75th birthday in Glasgow in June 2025 for helpful comments. Special thanks to Felix Brandt for valuable pointers to the literature.

1. Introduction

At the heart of social choice theory lies Arrow's (1951/63) result about the impossibility of independent preference aggregation. Indeed, it is fair to say that his formulation of the problem of collective choice as one of aggregating individual preference orderings into a collective one has shaped most of the social choice literature and the way researchers have approached the problem. However, it is also widely recognized that, ultimately, preferences represent choices. In fact, Arrow acknowledged himself that his assumption of a complete and transitive collective preference is a strong one, and that it would be sufficient that a group be able to make collective *choices*. As Arrow (1959) later showed himself, positing a complete and transitive ordering is tantamount to imposing the Weak Axiom of Revealed Preference (WARP) on collective choice behavior.

Following in his steps, various authors have shown that limited possibilities do emerge under less demanding impositions on collective choices (see, e.g., Gibbard, 1969/2014, Sen, 1969, Mas-Colell and Sonnenschein, 1972). These contributions have, however, maintained the assumption that collective choices are *binary*, that is, rationalizable by some collective preference. In this case, independence can be formulated as demanding that the collective preference between any two alternatives x and y be unaffected by how voters rank alternatives other than x and y. We refer to this condition as *binary Independence*. In contrast, Arrow's original notion relies only on the existence of a social choice function. It states that what society chooses from any given menu of alternatives be unaltered whenever two profiles of individual preferences agree in terms of every individual's ranking of alternatives in that menu. To be precise, the original condition reads (Arrow, 1951, Condition 3):[1]

Arrow Independence: Let $R_1, \ldots, R_n$ and $R'_1, \ldots, R'_n$ be two sets of individual orderings and let $c(A)$ and $c'(A)$ be the corresponding social choice functions. If, for all individuals i and all x and y in a given environment A, xR_iy if and only if xR'_iy, then $c(A)$ and $c'(A)$ are the same.

If collective choices are binary, binary Independence appears to be the weaker condition as it restricts Arrow Independence to binary menus. Since all choice behavior is reducible to choices from binary menus in this case, however, Arrow Independence is also no stronger than binary Independence. For non-binary collective choices, it is *prima facie* unclear what collective "preference" binary Independence would make reference to. Of

[1] To be consistent with our notation below, we have changed Arrow's notation for menus from S to A and for the collective choice function from uppercase C to lowercase c in the following passage.

course, it is always possible to *define* a social preference via the base relation that is given by $xRy :\iff x \in c(\{x, y\})$. However, the base relation will, in general, not be interpretable as a revealed preference that generates choices. Indeed, this is the case (if and) only if choices are binary.

Considered in isolation, Arrow's original notion of independence is weak. Thus, it is the imposition of binariness, not independence *per se*, that severely restricts the space for possibilities. To see this, it is helpful to recapitulate that a choice function is binary if and only if it satisfies two basic conditions on contraction and expansion consistency across different menus of alternatives, also known as conditions α and γ (see, for example, Sen, 1969). Condition α requires that if an alternative x is chosen from some menu A, then so it be chosen from any sub-menu that contains x. Condition γ demands that if x is chosen from menu A as well as from B, then so it be chosen from their union (see further below for precise statements). To illustrate, consider the Borda method (or, indeed, any other scoring rule) for aggregating individual preferences. Conventionally, it proceeds by calculating, locally at every menu, the Borda ranking of all the alternatives in that menu and choosing the alternative(s) that come(s) out on top. In this local version, Borda is an independent method in Arrow's sense, as the collective choice from any given menu does not depend on how individuals rank alternatives outside of it. Yet collective choices may not be binary. For example, consider a universal set of three alternatives $X = \{x, y, z\}$. If we have $x \succ y \succ z$ and $z \succ x \succ y$ for two voters each, and $y \succ z \succ x$ for three voters, then $c(\{x, y, z\}) = y$ while $c(\{x, y\}) = x$. Thus, society's choice between the two alternatives x and y changes as a third (irrelevant) alternative becomes available. This marks a failure of contraction consistency (condition α), a condition on choice *across* menus. In contrast, Arrow's original notion of independence is a condition on choice *from a given* menu. Thus, Borda understood in this local fashion does *not* violate Arrow independence.[2] We emphasize this point, as there seems to have been some confusion in the literature starting with Arrow himself.[3]

[2] Alternatively, if society chooses from menus according to the global Borda ranking over the universal set X, then collective choices are binary (by construction) but violate Arrow independence, seeing that whether x or y are chosen from the menu $\{x, y\}$ now depends on how voters rank them relative to z. So, for example, we obtain $c(\{x, y\}) = y$ for the preference profile introduced above, while we conclude $c(\{x, y\}) = x$ if all three voters with $y \succ z \succ x$ have $y \succ x \succ z$ instead. For an early and lucid account of this observation, see Sen (1977, p 78f.). Sen refers to the local and global variants as the narrow and broad Borda rule respectively. For a recent discussion, see also Brandl and Brandt (2020, p. 816). We thank Felix Brandt for pointing us to this literature.

[3] See the remarks following the statement of Condition 3 (the independence condition) in Arrow (1963, p.27): "The reasonableness of this condition can be seen by consideration of the possible results in a method of choice which does not satisfy Condition 3, the rank-order method of voting frequently used in clubs. [...] In particular, suppose that there are three voters and four candidates, x, y, z, and w.

In this paper, we propose to reappraise the Arrow problem in the framework of aggregating choice functions. This allows us to capture the various relaxations of collective rationality studied in the literature in a unified framework and to go beyond. Crucially, we allow for less then perfectly rational (and potentially non-binary) choice not only at the collective but also at the individual level. Assuming the same choice consistency conditions at the individual and the collective level allows us to tractably study the problem within the general *judgement aggregation* model developed, among others, by List and Pettit (2002), Dietrich (2007), Dietrich and List (2007), Nehring and Puppe (2002, 2010) and Dokow and Holzman (2010). Within this model, a choice function can be described by specifying, for each menu A and each alternative $x \in A$, if x is choosable from A or not. This formulation suggests a natural strengthening of independence in the Arrovian spirit: that the collective decision whether or not x is choosable from menu A should depend only on the individual views about *this* issue.

Evidently and unsurprisingly, the stronger independence condition cannot help escape the Arrow impossibility. Indeed, our first main result shows that even on the domain of all binary choice functions aggregation is (monotone) independent if and only if it is dictatorial. Strikingly, this result is disproportionately driven by expansion consistency (condition γ). If there at least four alternatives, imposing γ alone implies dictatorial social choices for all menus except the universal set. In contrast, when giving up on condition γ, possibilities emerge. On the space of path-independent choice functions (satisfying α as well as the *Aizerman* condition, an alternative notion of expansion consistency), non-dictatorial aggregation is possible, although necessarily in the form of oligarchies. Path-independent choice functions can be rationalized in terms of the maximal elements of a finite collection of linear orderings (Aizerman and Malishevski, 1981). Interpreting these orderings as representing the 'multiple selves' of the decision-maker, our result implies that all independent aggregation rules can be simply described as taking the collection of the oligarchs' individual multiple selves as the multiple 'selves' of the group. When dropping consistency conditions altogether, quota-rules emerge

Let the weights for the first, second, third, and fourth choices be 4, 3, 2, and 1, respectively. Suppose that individuals 1 and 2 rank the candidates in the order x, y, z, and w, while individual 3 ranks them in the order z, w, x, and y. Under the given electoral system, x is chosen. Then, certainly, if y is deleted from the ranks of the candidates, the system applied to the remaining candidates should yield the same result, especially since, in this case, y is inferior to x according to the tastes of every individual; but, if y is in fact deleted, the indicated electoral system would yield a tie between x and z." Yet, from the perspective of independence, there is nothing suspect about this, seeing that if y is removed, then the menu of alternatives to be chosen from is *not* the same as before. What Arrow describes here is a violation of WARP (specifically, the strong expansion property β). Presumably, the confusion surrounding this has been helped further by the unfortunate convention by which parts of the literature refer to WARP, or variants thereof, as 'Independence of Irrelevant Alternatives'.

as the (monotone) independent and anonymous aggregation methods that satisfy an additional weak neutrality condition. (Imposing α is still possible but restricts the set of consistent quotas.) We note that both our stronger notion of independence and our imposition of the same consistency conditions at the individual and collective levels strengthens our possibility results while making our impossibility results weaker. At the end of the paper, we briefly discuss a weaker notion of independence (which is still stronger than Arrow's original notion) and note that further possibilities arise in the form of approval voting.

To the best of our knowledge, there are only few contributions in the literature that explicitly addresses the aggregation of choice functions. The ones we are aware of are Aizerman and Aleskerov (1986), Sandroni and Sandroni (2021), Shelah (2005) and Sen (1993). Shelah (2005) provides a very general impossibility result on the aggregation of singleton-valued choice functions under a symmetry condition but without any further rationality assumptions. However, this intriguing result makes crucial use of the assumption that choices are singleton-valued, which we do not assume here. (For recent work in this same direction, see Roy and Sadhukhan (2025).) Sen (1993) derives an impossibility result without imposing any rationality requirements at the collective level. He formalizes independence in terms of decisiveness over pairs of alternatives that allow groups of individuals to block an alternative when the other one is present no matter what the menu. In our view, this is both, a long way conceptually from Arrow's original notion, and it re-introduces an element of context independence that the non-insistence on consistency requirements was meant to do away with. The two contributions most closely related to our approach are Aizerman and Aleskerov (1986) and Sandroni and Sandroni (2021). The former authors use a similar independence condition than we do, which they refer to as 'locality,' but they allow for empty choice sets and axiomatically characterize specific aggregation methods. The latter authors significantly weaken the independence condition and show that non-dictatorial aggregation becomes possible once full rationality in the sense of WARP is relaxed; however, these possibilities are not particularly attractive since they are still 'almost' dictatorial.

The rest of this paper is structured as follows. Section 2 introduces our model, recapitulates important results about the rationalizablity of choice functions, and presents our notion of independence. Section 3 presents our main results. Section 4 discusses the results in light of our independence condition and proposes a weakening that is intermediate in strength between the former and Arrow's original notion.

2. The Model

Let X be a (possibly infinite) set of alternatives $x, y, z \in X$ such that $|X| \geq 3$. Denote by $\mathcal{A} = 2^X \backslash \{\emptyset\}$ the collection of all non-empty decision problems (menus) $A \in \mathcal{A}$. A choice function (CF) is a mapping $c : \mathcal{A} \rightrightarrows X$ such that for all $A \in \mathcal{A}$, $c(A) \subseteq A$. Note that, in general, this allows for choice sets $c(A)$ to contain multiple alternatives.[4]

Let $\mathcal{C} = \{c : c \text{ is a CF}\}$ be the collection of all choice functions (on X) and $n \in \mathbb{N}$, $n \geq 2$, be the number of individuals in a group/society. An aggregation function $f : \mathcal{C}^n \to \mathcal{C}$ maps every *profile* of choice functions $(c_1, \ldots, c_n)$ into a social choice function $c = f(c_1, \ldots, c_n)$. We say that f is *consistent* on a *domain* $\emptyset \neq \mathcal{D} \subseteq \mathcal{C}$ if $f(\mathcal{D}^n) \subseteq \mathcal{D}$.[5] Below, we consider aggregation on (sub-)domains $\mathcal{D}$ obtained by imposing (consistency) conditions on the choice functions under consideration.

As a minimal restriction, we demand that choice sets be non-empty when choosing from finite menus.

Finite Non-Emptiness:

$$|A| < \infty \implies c(A) \neq \emptyset. \tag{FNE}$$

We denote by $\mathcal{C}_{\text{fne}} = \{c \in \mathcal{C} : c \text{ satisfies (FNE)}\}$ the collection of all choice functions satisfying finite non-emptiness.

2.1. Rationalizable Choice Functions

Of particular interest in economics are choice functions that are *rationalizable* as maximal elements of some relation(s) $R \subseteq X \times X$. For any such R, call P its asymmetric component (that is $xPy \iff (xRy \& \neg yRx)$) and I its symmetric component ($xIy \iff (xRy \& yRx)$). We say that R is:

1. *acyclic* if P does not contain a cycle;
2. *quasi-transitive* if P is transitive;
3. a *weak order* if R is complete and transitive;
4. a *linear order* if R is an asymmetric weak order.

[4] Some authors refer to the general concept as a 'choice correspondence' and reserve the name 'choice function' for choice correspondences that contain (at most) a single alternative.

[5] In other words, when studying consistent aggregation functions on $\mathcal{D}$, it is without loss of generality to restrict the *co-domain* to $\mathcal{D}$ as well. Therefore, for the rest of this paper, we simply refer to $\mathcal{D}$ as the domain.

Note that these properties are ordered from weak to strong. That is, every linear order is a weak order, every weak order is quasi-transitive and every quasi-transitive R is acyclic.[6] The latter is a minimal condition if R is to rationalize a choice function $c \in \mathcal{C}_{\text{fne}}$ in terms of maximal elements seeing that $\max_A R := \{x \in A : yPx \text{ for no } y \in A\}$ is non-empty for all finite A if and only if R is acyclic.

A by now extensive literature on choice theory has characterized notions of rationalizability in terms of consistency requirements on choice functions. As a widely known example of such a result, recall that a choice function is rationalized by some weak order if and only if it satisfies the *Weak Axioms of Revealed Preference* (WARP). We list some of the most important and well-known conditions on choice functions here.

WEAK AXIOM OF REVEALED PREFERENCE (WARP):

$$\text{if } \{x, y\} \subseteq A \cap B, \text{ then } x \in c(A) \implies (y \in c(B) \implies x \in c(B)). \tag{WARP}$$

CONTRACTION (α/CHERNOFF):

$$c(A \cup B) \cap A \subseteq c(A). \tag{α}$$

STRONG EXPANSION (β):

$$c(A \cup B) \cap A \neq \emptyset \implies c(A) \subseteq c(A \cup B). \tag{β}$$

EXPANSION (γ):

$$c(A) \cap c(B) \subseteq c(A \cup B). \tag{γ}$$

AIZERMAN(-EXPANSION):

$$c(A \cup B) \subseteq A \implies c(A) \subseteq c(A \cup B). \tag{Aiz}$$

PATH-INDEPENDENCE:

$$c(A \cup B) = c(c(A) \cup B). \tag{PI}$$

In general, WARP is equivalent to CONTRACTION (α) and STRONG EXPANSION (β). An interesting sub-case arises when c is singleton-valued (i.e., $|c(A)| = 1$ for all non-empty $c(A) \neq \emptyset$). In this case, WARP reduces to CONTRACTION (α). Moreover, STRONG EXPANSION (β) is stronger than both EXPANSION (γ) and AIZERMAN. We

[6] As an example of an acyclic relation that is not quasi-transitive, consider $R \subset \{x, y, z\}$ such that $xPyPzIx$. On the other hand, $zIyIxPy$ is both acyclic and quasi-transitive (but not a weak/linear order).

summarize this in the following lemma. A proof can be found, for example, in Moulin (1985).

Lemma 1. *Let $c \in \mathcal{C}_{fne}$.*

1. *c satisfies* (WARP) *if and only if it satisfies* (α) *and* (β)*.*
2. *c satisfies* (β) *only if it satisfies* (γ) *and* (Aiz)*.*
3. *If $|c(A)| \leq 1$ for all $A \in \mathcal{A}$, then c satisfies* (α) *only if it satisfies* (β)*.*
4. *c satisfies* (PI) *if and only if it satisfies* (α) *and* (Aiz)*.*

Based on Moulin (1985), the following lemma summarizes important results about the rationalizability of choice functions in the literature.

Lemma 2. *Let $c \in \mathcal{C}_{fne}$.*

1. *If and only if c satisfies* (α) *it is sub-rationalizable by some linear order $R \subseteq X \times X$: $\max_A R \subseteq c(A)$ for all $A \in \mathcal{A}$.*
2. *If and only if c satisfies* (α) *and* (Aiz) *it is pseudo-rationalizable: there exist linear orders $R_1, \ldots, R_k \subseteq X \times X$ such that $c(A) = \bigcup_{j=1,\ldots,k} \max_A R_j$ for all $A \in \mathcal{A}$.*
3. *If and only if c satisfies* (α) *and* (γ) *it is binary, that is, rationalizable by some complete and acyclic binary relation $R \subseteq X \times X$: $c(A) = \max_A R$ for all $A \in \mathcal{A}$.*
4. *If and only if c satisfies* (α), (γ) *and* (Aiz) *it is rationalizable by some complete and quasi-transitive $R \subseteq X \times X$: $c(A) = \max_A R$ for all $A \in \mathcal{A}$.*
5. *If and only if c satisfies* (WARP) *it is rationalizable by some weak order $R \subseteq X \times X$: $c(A) = \max_A R$ for all $A \in \mathcal{A}$. If c is singleton-valued, weak order can be replaced by linear order in this statement.*

The above results inform the following definitions:

- $\mathcal{C}_{\text{sub}} := \{c \in \mathcal{C}_{\text{fne}} : c \text{ is sub-rationalizable}\}$,
- $\mathcal{C}_{\text{psd}} := \{c \in \mathcal{C}_{\text{fne}} : c \text{ is pseudo-rationalizable}\}$,
- $\mathcal{C}_{\text{bin}} := \{c \in \mathcal{C}_{\text{fne}} : c \text{ is rationalizable by some ayclic } R \subseteq X \times X\}$,
- $\mathcal{C}_{\text{qua}} := \{c \in \mathcal{C}_{\text{fne}} : c \text{ is rationalizable by some quasi-transitive } R \subseteq X \times X\}$,

- $\mathcal{C}_{\text{wo}} := \{c \in \mathcal{C}_{\text{fne}} : c$ is rationalizable by some weak order $R \subseteq X \times X\}$,
- $\mathcal{C}_{\text{lo}} := \{c \in \mathcal{C}_{\text{fne}} : c$ is rationalizable by some linear order $R \subseteq X \times X\}$.

Note that $\mathcal{C}_{\text{lo}} \subseteq \mathcal{C}_{\text{wo}} \subseteq \mathcal{C}_{\text{qua}} \subseteq \mathcal{C}_{\text{psd}}, \mathcal{C}_{\text{bin}} \subseteq \mathcal{C}_{\text{sub}} \subseteq \mathcal{C}_{\text{fne}}$.

2.2. Collective Choice and Aggregation of Elementary Choice Judgments

When aggregating a profile of choice functions into a collective choice function, society is faced, for every menu $A \in \mathcal{A}$, $|A| \geq 2$ and every alternative $x \in A$, with the issue of whether it should (collectively) choose x from A.[7] More generally, both at the individual and at the collective level, a choice function may be thought of as a complete set of such elementary choice judgments $x \in c(A)$.

Imposing consistency conditions on choice functions then translates into restrictions on what sets of judgments are considered feasible. For example, imposing condition (α) is inconsistent with simultaneously judging $x \in c(A)$ to be true and $x \in c(B)$ to be false when $x \in B \subset A$. Thus, consistency conditions on choice functions translate into logical dependencies (entailments) between issues. This approach allows us to employ the machinery developed in Nehring and Puppe (2002, 2010) to study what domains $\mathcal{D}$ entail dictatorial aggregation rules and when and what kind of non-dictatorial aggregation is possible.

Modeling choice functions as complete sets of (elementary choice) judgments suggests a natural notion of independence. We say that an aggregation rule is independent if the collective decision on whether some $x \in A$ is chosen from A may only depend on individual judgments on *this* issue alone. In other words, if two profiles of individual choice functions agree in terms of individual judgments on $x \in c_i(A)$, for all $i = 1, \dots, n$, then the collective decision on whether $x \in c(A)$ must be the same for both profiles.

Independence: Consider any $A \in \mathcal{A}$ and $x \in A$. Let $c = f(c_1, \dots, c_n)$ and $c' = f(c'_1, \dots, c'_n)$. If, for all $i = 1, \dots, n$, $x \in c_i(A) \iff x \in c'_i(A)$, then $x \in c(A) \iff x \in c'(A)$.

As such, independence may be seen as a condition of informational parsimony stipulating that collective choices can only rely on information that is pertinent to the choice to be made. Our notion of independence is inspired by viewing social choice as the aggregation of elementary choice judgments $x \in c(A)$ and demands that the collective

[7] If A is a singleton, say $A = \{x\}$, the issue is trivial seeing that x must be chosen given (FNE).

choosability of some alternative x in menu A be decided solely based on the configuration of individuals' choice judgments on this issue. In contrast, Arrow's original version requires that what is chosen from a menu may depend on how individuals rank the alternatives in it. In choice functional terms, this corresponds to allowing the collective decision to depend on individual choice behavior for all *submenus* of A.[8] Thus, our notion of independence is stronger than Arrow's. At the end of the paper, we discuss an intermediate notion – which we call *independence across menus* – that demands that what is chosen collectively from some menu must not depend on individual choices from other menus. While this allows for dependence within menus, it excludes reliance on further positional information.

In the presence of independence, it is natural to demand that aggregation happens in a monotone fashion. That is to say that increased support among all individuals in favor of some choice $x \in c(A)$ should lead to it being chosen collectively if it was before. Vice versa, if more individuals reject $x \in c(A)$, x must not be chosen collectively if it wasn't before. Thus, we strengthen independence to the following condition.

Monotone Independence: Consider any $A \in \mathcal{A}$ and $x \in A$. Let $c = f(c_1, \dots, c_n)$ and $c' = f(c'_1, \dots, c'_n)$. (i) If, for all $i = 1, \dots, n$, $x \in c_i(A) \implies x \in c'_i(A)$, then $x \in c(A) \implies x \in c'(A)$; (ii) if for all $i = 1, \dots, n$, $x \notin c_i(A) \implies x \notin c'_i(A)$, then $x \notin c(A) \implies x \notin c'(A)$.

As an example of a monotone and independent aggregation rule consider the social choice function resulting from *majority voting* on all issues. That is, we define $f_{maj} : \mathcal{C}^n \to \mathcal{C}$ such that, for all $A \in \mathcal{A}$, $x \in f_{maj}(c_1, \dots, c_n)(A) := c_{maj}(A) \iff \frac{1}{n}|\{i : x \in c_i(A)\}| \geq n/2$.[9] However, as we note below, majority voting is inconsistent in general, even when only imposing (FNE).

In analogy to Arrow's weak Pareto condition, we impose a weak unanimity assumption.

Unanimity: For all $c \in \mathcal{C}^n$, $f(c, \dots, c) = c$.

Unanimity only requires that if all individual choice functions are the same, this be the collective choice function. Note, however, that in the presence of monotone independence it is equivalent to the stronger notion of *issue-wise* unanimity. That is, for all $A \in \mathcal{A}$ and all $x \in A$, if $x \in c_i(A)$ for all $i = 1, \dots, n$, then $x \in c(A) = f(c_1, \dots, c_n)(A)$.

[8] Note that if individual choice functions are binary, the underlying preference orderings are revealed by the entirety of choices from all these submenus. (In fact, looking at all binary submenus would be sufficient.)

[9] Note that, given this definition, we break ties in favor of inclusion of alternatives in the collective choice set.

We call an aggregation function satisfying unanimity and monotone independence an *Arrovian* aggregator.

3. (Im)Possibility of Arrovian Aggregation

Here we address the question of whether consistent monotone independent aggregation is possible on the different domains $\mathcal{D} \subseteq \mathcal{C}_{\text{fne}}$ defined above. Note that as the domain $\mathcal{D}$ is restricted further, aggregation needs to be consistent for a smaller set of individual choice functions (as more requirements are imposed) but, at the same time, needs to satisfy more stringent consistency conditions at the collective level.

From a majoritarian perspective, the space of possibilities is limited from the outset by the observation that ($x \in c(A)$-wise) majority voting is inconsistent on $\mathcal{D} = \mathcal{C}_{\text{fne}}$ (i.e. even without imposing further restrictions on choice functions) except for the special cases of three alternatives and two or four individuals.

Proposition 1. *Let $|X| \geq 3$. f_{maj} is consistent on $\mathcal{D} = \mathcal{C}_{fne}$ if and only if $|X| = 3$ and $n \in \{2, 4\}$.*

For example, consider the case of $n = 3$ individuals and let $A = \{x, y, z\} \subseteq X$. Consider a profile of choice functions $c_1, c_2, c_3 \in \mathcal{C}_{\text{fne}}$ such that $c_1(A) = \{a\}$, $c_2(A) = \{b\}$ and $c_3(A) = \{c\}$. Note that $f_{maj}(A) = \emptyset$. Thus, $c_{maj} = f_{maj}(c_1, c_2, c_3)$ does not satisfy (FNE), i.e., $f_{maj}(c_1, c_2, c_3) \notin \mathcal{C}_{\text{fne}}$.

When choice functions are rationalizable by some acyclic relation (satisfy contraction property (α) and extension property (γ)) all consistent Arrovian aggregators are necessarily dictatorial. The same holds when more (stringent) consistency conditions are imposed (quasi-transitive, weak or linear order rationalizable choice functions).

Theorem 1. *An Arrovian aggregation rule f is consistent on $\mathcal{D} = \mathcal{C}_{bin}$ if and only if it is a dictatorship; that is, there is some $j \in \{1, \ldots, n\}$ such that for all $(c_1, \ldots, c_n) \in \mathcal{C}_{bin}^n$:*

$$f(c_1, \ldots, c_n) = c_j.$$

The same holds for $\mathcal{D} = \mathcal{C}_{qua}, \mathcal{C}_{wo}, \mathcal{C}_{lo}$.

Strikingly, this result is disproportionately driven by the expansion property (γ). Indeed, imposing (γ) alone (alongside (FNE)) implies dictatorial social choice from all menus that are not the universal set X, granted that X contains at least four alternatives. On the universal set, the 'dictator' can force the choice of any alternative but cannot necessarily veto against it.

Theorem 2. *For $|X| \geq 4$, an Arrovian aggregation rule f is consistent on $\mathcal{D} = \{c \in \mathcal{C}_{fne} : c$ satisfies $(\gamma)\}$ if and only if there is some $j \in \{1, \ldots, n\}$ such that for all $(c_1, \ldots, c_n) \in \mathcal{D}^n$:*

(i) for all $A \in \mathcal{A} \setminus \{X\}$: $f(c_1, \ldots, c_n)(A) = c_j(A)$,

(ii) $f(c_1, \ldots, c_n)(X) \supseteq c_j(X)$.

On the other hand, when considering extension property (Aiz) instead of (γ) alongside (α), i.e. for path-independent choice functions pseudo-rationalizable by multiple linear orders ('selves'), possibilities emerge in the form of oligarchic rules.

Theorem 3. *An Arrovian aggregation rule f is consistent on $\mathcal{D} = \mathcal{C}_{psd}$ if and only if it is an oligarchic choice rule; that is, there exists some $\emptyset \neq M \subseteq \{1, \ldots, n\}$ such that, for all $(c_1, \ldots, c_n) \in \mathcal{C}_{psd}^n$ and all $A \in \mathcal{A}$:*

$$f(c_1, \ldots, c_n)(A) = \bigcup_{i \in M} c_i(A).$$

The only anonymous oligarchic rule (obtained by letting $M = N$) is the issue-wise unanimity rule which, for all $A \in \mathcal{A}$ and all $x \in A$, declares x as choosable from A unless x is unanimously rejected by all individuals. Interestingly, the collective choices under this unanimity rule coincide with the collective choice function that is pseudo-rationalized by the collection of all 'selves' in society.[10]

Two important remarks are in order. First, the unanimity rule is a (anonymous) Arrovian procedure for all domains $\mathcal{D}$ we consider here. Yet as Theorem 1 shows, it is not consistent on $\mathcal{D} = \mathcal{C}_{\text{bin}}, \mathcal{C}_{\text{qua}}, \mathcal{C}_{\text{wo}}, \mathcal{C}_{\text{lo}}$. This is due to the fact that collective choices under the unanimity rule violate property (γ). For example, consider $X = \{a, b, c\}$ and two individuals with $a \succ_1 b \succ_1 c$ and $c \succ_2 b \succ_2 a$. Unanimity rule yields $c(\{a, b\}) = \{a, b\}$ and $c(\{b, c\}) = \{b, c\}$ as in both cases there is one voter choosing each of the alternatives. Consequently, $b \in c(\{a, b\}) \cap c(\{b, c\})$. At the same time, $b \notin \{a, c\} = c(\{a, b, c\})$ as no voter chooses b from $\{a, b, c\}$, in violation of property (γ).[11]

Second, the strong Pareto rule, according to which all alternatives are choosable from a given menu that are not strongly Pareto-dominated by some other alternative (i.e., strictly preferred by all voters) in it, is known to yield a binary collective choice function

[10] If all c_i are pseudo-rationalizable by $R_1^i, \ldots, R_{k_i}^i$, the collection of all 'selves' in society $\bigcup_{i=1,\ldots,n} \{R_1^i, \ldots, R_{k_i}^i\}$ allows one to define a pseudo-rationalizable collective choice function.

[11] Similar arguments show that collective choices under any oligarchic rule violate (γ).

on $\mathcal{D} = \mathcal{C}_{\text{bin}}$ (that is, choices are rationalized by an acyclic collective preference relation).[12] However, it is not an Arrovian procedure in our model as it violates independence. To see this, reconsider our example for which $a \succ_1 b \succ_1 c$ and $c \succ_2 b \succ_2 a$. As no alternative is strongly Pareto-dominated in $\{a, b, c\}$, we have $b \in \{a, b, c\} = c(\{a, b, c\})$, although $b \notin c_1(\{a, b, c\})$ and $b \notin c_2(\{a, b, c\})$.[13] Now suppose the preference of the second individual changes so that $c \succ_2' a \succ_2' b$. As a strongly Pareto-dominates b we have $b \notin c(\{a, b, c\})$. However, the second individual has not actually changed her choice behavior for b in the menu $\{a, b, c\}$ seeing that still $b \notin c_2'(\{a, b, c\})$.[14] Thus, the strong Pareto rule, just as well as other 'oligarchic' rules in the sense of Gibbard (1969/2014), are not independent in our choice-functional setting. These findings cast some doubt on their status in the literature as representing 'possibilities' for independent aggregation.

We now restrict attention to finite X and consider the possibility of non-oligarchic rules (in which no single individual possesses veto power) when dropping the extension properties (γ)/(Aiz). We focus on rules that treat all individuals equally. Thus, the following condition is natural.

Anonymity: Let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation (of individuals). Then $f(c_1, \ldots, c_n) = f(c_{\pi(1)}, \ldots, c_{\pi(n)})$.

Anonymity requires that 'voter's names do not matter' in the sense that the collective choice rule be invariant to permuting (re-labeling) all individuals. Moreover, we impose a parallel condition on alternatives within the *same* menu.

Menu-level Neutrality: Consider any $A \in \mathcal{A}$ such that $A = \{x_1, \ldots, x_m\}$ and any permutation $\pi : \{1, \ldots, m\} \to \{1, \ldots, m\}$. Let $c = f(c_1, \ldots, c_n)$ and $c' = f(c_1', \ldots, c_n')$. If, for all $i = 1, \ldots, n$ and all $j = 1, \ldots, m$, $x_j \in c_i(A) \iff x_{\pi(j)} \in c_i'(A)$, then for all $j = 1, \ldots, m$, $x_j \in c(A) \iff x_{\pi(j)} \in c'(A)$.

Menu-level neutrality requires that all alternatives in any given menu are treated equal ('neutral') by the aggregation rule in the sense that permuting them results in choosing

[12] Note that, if individual preferences are quasi-transitive, the strong Pareto rule yields quasi-transitive social choices (that is, it is consistent on $\mathcal{C}_{\text{qua}}$). If individual preferences are weak orders, the weak Pareto rule according to which all alternatives are choosable that are not weakly Pareto dominated by some other alternative (i.e., weakly preferred by all voters and strictly preferred by some voter), yields quasi-transitive collective choices as well.

[13] In particular, the strong Pareto rule violates issue-wise unanimity and is not a unanimity rule in our more general choice function framework. The same is true for the weak Pareto rule.

[14] Indeed, the second individual has not changed her overall choice behavior from $\{a, b, c\}$ as $\{c\} = c_2'(\{a, b, c\}) = c_2(\{a, b, c\})$. Thus, the strong Pareto rule even violates the weaker notion of *independence across menus* that we introduce further below. Again, the same observations apply to the weak Pareto rule.

exactly the permuted originally chosen alternatives at the collective level. In other words, collective choice from any menu is invariant under a re-labeling of the alternatives. Taken together with monotone independence, this assumption implies that, at a given menu $A \in \mathcal{A}$, the rule determining whether any $x \in A$ be chosen collectively is the *same* for all alternatives in A.

Taken together, unanimity, monotone independence and anonymity require that aggregation, at any issue $x \in c(A)$, happens by setting an acceptance *quota* $0 < q_{x\in c(A)} < 1$ such that $x \in c(A) \iff \frac{1}{n}|\{i \in \{1, \dots, n\} : x \in c_i(A)\}| \geq q_{x\in c(A)}$. Whenever some coalition $W \subseteq \{1, \dots, n\}$ of individuals exceeds the quota $q_{x\in c(A)}$, we say that W is a *winning coalition* (for $x \in c(A)$). Given menu-level neutrality these quotas must be 'effectively equal' for all alternatives in some given menu in the sense that they imply the same winning coalitions for all $x \in A$.[15]

Theorem 4. *An Arrovian, menu-level neutral and anonymous aggregation rule f is consistent on $\mathcal{C}_{fne}$ if and only if it is a quota-rule such that, for all $A \in \mathcal{A}$ with $|A| \geq 2$, the quotas for collectively choosing any x from A imply the same structure of winning coalitions and are such that (i) if $n \leq |A|$, then $0 < q_{x\in c(A)} \leq 1/n$; (ii) if $n > |A|$, then $0 < q_{x\in c(A)} \leq \frac{1}{|A|}(1 - \frac{r}{n}) + \frac{1}{n}\mathbb{1}(r \neq 0)$ where $r = n \mod |A|$.*

Theorem 4 implies that (maximal) consistent acceptance quotas at menu A approach $1/|A|$ for large societies (as $n/|A|$ grows large) and need to decrease in the size of A. For menus $|A| \geq n$, consistent quota rules reduce to unanimity rule on A for which each individual can veto not choosing any $x \in A$ collectively (i.e. rejecting $x \in c(A)$ requires unanimous consent).

When also imposing contraction consistency (α), every winning coalition at some menu A needs to be winning at all sub-menus $B \subseteq A$ to ensure (α) holds at the collective level. This implies that consistent quotas need to 'effectively'[16] decrease when moving to sub-menus. As all menus are sub-menus of the universal set, maximal consistent quotas are thus determined by $|X|$. Indeed, Theorem 4 implies that if $q_{x\in c(A)} = \bar{q}$ for all $A \in \mathcal{A}$ and $x \in A$, then $0 < \bar{q} \leq 1/|X|$ for large societies (as $n \to \infty$).

4. Discussion and Outlook

The analysis of the previous section provides two main messages. First, non-dictatorial (monotone) independent aggregation, even in the stronger form we consider here, is

[15] Note that - due to integer effects - a whole interval of quotas can induce the same set of winning coalitions.

[16] That is, bar of any integer effects.

possible if one is willing to give up binariness of the social choice function. This is true even if individual choices are not themselves binary, just as long as they are path-independent. Second, if one insists on binary choice functions, (monotone) independent aggregation is necessarily dictatorial. Interestingly, the driving force behind this turns out to be the expansion consistency condition (γ). To see this, remember that binariness is equivalent to α and γ, while path-independence is equivalent to α and the Aizerman condition.

Yet, the independence assumption we impose is undoubtedly strong. It requires not only that aggregation is independent across different menus but also that it is independent across alternatives *within* any given menu. While we consider the former a natural condition of informational parsimony, the latter may be unnecessarily strong. Consider the following weakening of independence.

Independence Across Menus: Consider any $A \in \mathcal{A}$. Let $c = f(c_1, \ldots, c_n)$ and $c' = f(c'_1, \ldots, c'_n)$. If, for all $i = 1, \ldots, n$, $c_i(A) = c'_i(A)$, then $c(A) = c'(A)$.

Thus, if collective choices are independent across menus, determining the collective choice set for some menu A only involves eliciting individual choice sets for A. In some sense, this condition might be considered the most natural analogue to Arrow's notion in the more general setting of aggregating choice functions. While Arrow's original condition allows for the collective choice from some menu to depend on individual choice behavior from all submenus, it is not immediately clear why this would be relevant information if individual choices are (potentially) not binary.

As an example of an aggregation rule that satisfies independence across menus but not our fully-fledged notion of independence consider 'approval voting' which, for every menu A, collectively chooses all alternatives with maximal approval by individuals. That is, $f_{\mathrm{AV}}(c_1, \ldots, c_n)(A) = c_{\mathrm{AV}}(A) := \mathrm{argmax}_{x \in A} |\{i \in \{1, \ldots, n\} : x \in c_i(A)\}|$. On the other hand, Borda rule (more generally, all scoring rules) does not satisfy independence across menus (but satisfies Arrow independence) seeing that calculating (Borda) scores at some menu $A \in \mathcal{A}$ relies on positional information to be revealed by individual choices from any pair of alternatives $x, y \in A$.[17]

Observation 1. *f_{AV} satisfies Independence Across Menus. It is consistent on $\mathcal{D} = \mathcal{C}_{fne}$ but inconsistent on $\mathcal{D} = \mathcal{C}_{sub}, \mathcal{C}_{bin}, \mathcal{C}_{psd}, \mathcal{C}_{qua}, \mathcal{C}_{wo}, \mathcal{C}_{lo}$.*

[17] Note that on $\mathcal{D} = \mathcal{C}_{\mathrm{lo}}$, eliciting choices from all pairs of alternatives in some given $A \in \mathcal{A}$ reveals the individual's underlying linear order when restricted to A.

Observation 1 states that approval voting is not (sub-/pseudo-)rationalizable in any of the ways considered here. To see this, note that it fails to satisfy the contraction property (α) even if all individual choice functions do. Indeed, let $\{x, y, z\} = A \subseteq X$ and suppose that c_1, c_2, c_3 are rationalized by the linear orders $x \succ_1 y \succ_1 z$, $y \succ_2 x \succ_2 z$ and $z \succ_3 y \succ_3 x$ respectively. Thus, $c_1(A) = x$, $c_2(A) = y$, $c_3(A) = z$ and $c_1(\{x, y\}) = x$ but $c_2(\{x, y\}) = c_3(\{x, y\}) = y$. Consequently, $x \in A = c_{\mathrm{AV}}(A)$ but $x \notin \{y\} = c_{\mathrm{AV}}(\{x, y\})$ failing (α). Thus, Approval Voting does not guarantee any of the consistency properties considered in this paper at the collective level. Whether Approval Voting satisfies *any* consistency property known in the literature remains an open question. At the same time, Independence Across Menus defines the informational basis for Approval Voting. Thus, it might be possible to characterize the latter in terms of it and a suitably defined notion of positive responsiveness. We believe that this is an interesting avenue for future research.

References

Aizerman, M. and Aleskerov, F. (1986). Voting operators in the space of choice functions. *Mathematical Social Sciences*, 11:201–242.

Aizerman, M. A. and Malishevski, A. V. (1981). General theory of best variants choice: Some aspects. *IEEE Transaction on Automatic Control*, 26:1030–1040.

Arrow, K. J. (1951). *Social Choice and Individual Values.* Yale University Press, New Haven.

Arrow, K. J. (1959). Rational choice functions and orderings. *Economica*, 26:121–127.

Arrow, K. J. (1963). *Social Choice and Individual Values: Second Edition.* Yale University Press, New Haven.

Brandl, F. and Brandt, F. (2020). Arrovian aggregation of convex preferences. *Econometrica*, 88(2):799–844.

Dietrich, F. (2007). A generalised model of judgment aggregation. *Social Choice and Welfare*, 28:529–565.

Dietrich, F. and List, C. (2007). Arrow's theorem in judgment aggregation. *Social Choice and Welfare*, 29:19–33.

Dokow, E. and Holzman, R. (2010). Aggregation of binary evaluations. *Journal of Economic Theory*, 145:495–511.

Gibbard, A. F. (2014). Social choice and the arrow conditions. *Economics and Philosophy*, 30:269–284.

List, C. and Pettit, P. (2002). Aggregating sets of judgments: An impossibility result. *Economics and Philosophy*, 18:89–110.

Mas-Colell, A. and Sonnenschein, H. (1972). General possibility theorems for group decisions. *The Review of Economic Studies*, 39:185–192.

Moulin, H. (1985). Choice functions over a finite set: A summary. *Social Choice and Welfare*, 2:147–160.

Nehring, K. (2006). Oligarchies in judgment aggregation. *Working Paper.*

Nehring, K. and Puppe, C. (2002). Strategy-proof social choice on single-peaked domains. *Working Paper.*

Nehring, K. and Puppe, C. (2010). Abstract arrowian aggregation. *Journal of Economic Theory*, 145:467–494.

Roy, S. and Sadhukhan, S. (2025). Aggregation of choice functions on non-rational domains. *Working Paper.*

Sandroni, A. and Sandroni, A. (2021). A comment on Arrow's impossibility theorem. *The B.E. Journal of Theoretical Economics*, 21:347–354.

Sen, A. (1969). Quasi-transitivity, rational choice and collective decisions. *The Review of Economic Studies*, 36:381–393.

Sen, A. (1977). Social choice theory: A re-examination. *Econometrica: journal of the Econometric Society*, pages 53–89.

Sen, A. (1993). Internal consistency of choice. *Econometrica*, 61:495–521.

Shelah, S. (2005). On the arrow property. *Working Paper.*

Appendix

We define $\mathcal{A}_{\geq 2} := \{A \in \mathcal{A} : |A| \geq 2\}$.

A. Proofs for Section 3

Proof of Proposition 1

We start with the special case that $|X| = 3$ and $n \in \{2, 4\}$. Consider menus of two alternatives first. As all individuals choose at least one alternative from them, there must be a weak majority for at least one alternative. Now consider the universal menu X. If $n = 2$, every single individual forms a weak majority by herself. If $n = 4$, at least one alternative must be chosen be chosen by two (or more) individuals forming a weak majority. Thus, all collective choice sets are non-empty.

Now let $|X| \geq 3$ and consider some menu $\{x, y, z\} := A \in \mathcal{A}_{\geq 2}$. The case of $n = 3$ individuals is covered in the main text. Let $n \geq 5$ and $k = n \mod 3$. If $k = 0$, let $(c_1, \ldots, c_n) \in \mathcal{C}^n_{\text{fne}}$ be such that $c_i(A) = x$, $c_i(A) = y$ and $c_i(A) = z$ for $n/3$ individuals each. If $k = 1$, let $(c_1, \ldots, c_n) \in \mathcal{C}^n_{\text{fne}}$ be such that $c_i(A) = x$, $c_i(A) = y$ for $n/3$ individuals each and $c_i(A) = z$ for $(n-1)/3 + 1$ individuals. If $k = 2$, let $(c_1, \ldots, c_n) \in \mathcal{C}^n_{\text{fne}}$ be such that $c_i(A) = x$ for $n/3$ individuals and $c_i(A) = y$ and $c_i(A) = z$ for $(n-2)/3 + 1$ individuals each. Note that the share of voters in support of each alternative in A is thus bounded by $\frac{\frac{n-1}{3}+1}{n} = \frac{1}{3} + \frac{2}{3n} \underbrace{\leq}_{n \geq 5} \frac{1}{3} + \frac{2}{15} = \frac{7}{15} < \frac{1}{2}$. Consequently, $c_{\text{maj}}(A) = \emptyset$.

Proof of Theorems 1–4

We analyze the aggregation of choice correspondences as a judgment aggregation problem on a property space. This methodology was developed in Nehring and Puppe (2002); Nehring (2006); Nehring and Puppe (2010). Our results are applications of the general characterization results obtained therein.

Let $\emptyset \neq \mathcal{D} \subseteq \mathcal{C}$ be a domain of choice correspondences such that for all $A \in \mathcal{A}_{\geq 2} := \{A \in \mathcal{A}, |A| \geq 2\}$ and $x \in A$ there exist $c, c' \in \mathcal{D}$ such that $x \in c(A)$ and $x \notin c(A)$.[18] For example, $\mathcal{D} = \mathcal{C}_{\text{fne}}$, $\mathcal{D} = \mathcal{C}_{\text{psd}}$ or $\mathcal{D} = \mathcal{C}_{\text{wo}}$. For all $A \in \mathcal{A}_{\geq 2}$ and all $x \in A$, define $H_{x|A} := \{c \in \mathcal{D} : x \in c(A)\}$ and $H^c_{x|A} := \mathcal{D} \backslash H_{x|A}$ (note that every issue $H_{x|A}, H^c_{x|A}$ partitions $\mathcal{D}$). Thus, $H_{x|A}$ corresponds to the *property* that x is chosen from A. Let

[18] This is a minimal richness condition requiring that no issue $x \in c(A)$ is pre-determined. However, this assumption is not crucial. Alternatively, we can simply ignore issues $x \in c(A)$ which *all* choice correspondences agree on.

$\mathcal{H} = \{H_{x|A}, H^c_{x|A} : x \in A \in \mathcal{A}_{\geq 2}\}$ be the collection of all properties. As every choice correspondence is identified with a *unique* combination of properties, $(\mathcal{D}, \mathcal{H})$ defines a *property space.*

We say that a family (of properties) $\mathcal{G} \subseteq \mathcal{H}$ is *inconsistent* if $\bigcap \mathcal{G} = \emptyset$ (*consistent* if $\bigcap \mathcal{G} \neq \emptyset$). $\mathcal{G}$ is *critical* if it is *minimally* inconsistent; that is, $\bigcap \mathcal{G} = \emptyset$ and for all $G \in \mathcal{G}$, $\bigcap(\mathcal{G}\backslash\{G\}) \neq \emptyset$. The critical families capture the dependency structure between properties. If $H, G^c \in \mathcal{G}$ and $\mathcal{G}$ is critical, then property H *conditionally entails* property G (seeing that – conditional on the other properties in the family – if $x \in H$, then $x \in G$); we write $H \geq_0 G$ and let $\geq$ denote the transitive closure of $\geq_0$ and let $\equiv$ be the symmetric part of $\geq$. We note that $\geq_0$ (thus, $\geq$) is complementation-adapted; that is, if $H \geq_0 G$, then $G^c \geq_0 H^c$. Moreover, as all families $\{H, H^c\}$ are trivially critical, $\geq_0$ (thus, $\geq$) is reflexive. Note that:

- (FNE) implies that for all $A \in \mathcal{A}_{\geq 2}$ with $A = \{x_1, \ldots, x_m\}$: $\{H^c_{x_1|A}, \ldots, H^c_{x_m|A}\}$ is critical.
- (α) implies that for all $A, B \in \mathcal{A}_{\geq 2}$ with $B \subsetneq A$ and all $x \in B$: $\{H_{x|A}, H^c_{x|B}\}$ is critical.
- (Aiz) implies that for all $A, B \in \mathcal{A}_{\geq 2}$ with $\emptyset \neq B\backslash A = \{y_1, \ldots, y_m\}$ and all $x \in A$: $\{H_{x|A}, H^c_{x|A\cup B}, H^c_{y_1|A\cup B}, \ldots, H^c_{y_m|A\cup B}\}$ is critical.
- (γ) implies that for all $A, B \in \mathcal{A}_{\geq 2}$ for which neither $A \subseteq B$ nor $B \subseteq A$ and for all $x \in A \cup B$: $\{H_{x|A}, H_{x|B}, H^c_{x|A\cup B}\}$ is critical.

Lemma 3. *Let $\emptyset \neq \mathcal{D} \subseteq \mathcal{C}$.*

1. *Suppose all $c \in \mathcal{D}$ satisfy* (FNE) *and* (α)*. Then for all $A, B \in \mathcal{A}_{\geq 2}$, $x \in A$, $y \in B$, $x \neq y$: $H^c_{x|A} \geq H_{y|B}$.*
2. *Suppose all $c \in \mathcal{D}$ satisfy (γ) and (α). Then for all $A, B \in \mathcal{A}_{\geq 2}$, $x \in A$, $H_{x|A} \geq H^c_{x|A}$.*
3. *Suppose all $c \in \mathcal{D}$ satisfy* (Aiz) *and* (α)*. Then for all $A, B \in \mathcal{A}_{\geq 2}$, $x \in A$ and $y \in B$: $H_{x|A} \geq H_{y|B}$. Thus, $H_{x|A} \equiv H_{y|B}$ and $H^c_{x|A} \equiv H^c_{y|B}$.*

Proof. 1. By (α), $H^c_{x|A} \geq_0 (H^c_{x|A\cup B})$. By (FNE), $H^c_{x|A\cup B} \geq_0 H_{y|A\cup B}$. Again, by (α), $H_{y|A\cup B} \geq_0 H_{y|B}$.

2. Suppose first that $A \neq X$. By (γ), $H_{x|A} \geq_0 H^c_{x|X\backslash A\cup x}$ (conditional on $H^c_{x|X}$). By (α), $H^c_{x|(X\backslash A)\cup\{x\}} \geq_0 H^c_{x|X}$. Again, by (γ), $H^c_{x|X} \geq_0 H^c_{x|A}$ (conditional on

$H_{x|(X\backslash A)\cup\{x\}}$). Now, if $A = X$, let $B \subsetneq X$. By (α) and what we just showed, $H_{x|X} \geq_0 H_{x|B} \geq H^c_{x|B} \geq_0 H^c_{x|X}$.

3. Suppose first that $y \in B\backslash A$. By (Aiz), $H_{x|A} \geq_0 H_{y|A\cup B}$ (conditional on $\{H^c_{y_1|A\cup B}, \ldots, H^c_{y_k|A\cup B}, H^c_{x|A\cup B}\}$ for $(B\backslash A)\backslash\{y\} = \{y_1, \ldots, y_k\}$). By ($\alpha$), $H_{y|A\cup B} \geq_0 H^c_{y|B}$. Else $y \in A\cap B$. Then, by (α), $H_{x|A} \geq_0 H_{x|A\backslash\{y\}}$ and, by (Aiz), $H_{x|A\backslash\{y\}} \geq_0 H_{y|A\cup B}$ (conditional on $\{H^c_{y_1|A\cup B}, \ldots, H^c_{y_k|A\cup B}, H^c_{x|A\cup B}\}$ for $B\backslash A = \{y_1, \ldots, y_k\}$). Now, again by ($\alpha$), $H_{y|A\cup B} \geq_0 H_{y|B}$.

   Consequently, $H_{x|A} \geq H_{y|B}$ and $H_{y|B} \geq H_{x|A}$; thus, $H_{x|A} \equiv H_{y|B}$. By complementation-adaptedness, $H^c_{x|A} \equiv H^c_{y|B}$.

□

Lemma 4. *Let $\emptyset \neq \mathcal{D} \subseteq \mathcal{C}$ and $|X| \geq 4$. Suppose all $c \in \mathcal{D}$ satisfy* (FNE) *and* (γ). *Then, for all $A, B \in \mathcal{A}_{\geq 2}$ with $A, B \neq X$ and all $x \in A$, $y \in B$: $H_{x|A} \geq H^c_{y|B}$.*

Proof. We distinguish the following four cases:

Case 1: $y \notin A$, $x \notin B$
Note that $x \neq y$. By condition (γ), we have $H_{x|A} \geq_0 H^c_{x|\{x,y\}}$. By condition (FNE), $H^c_{x|\{x,y\}} \geq_0 H_{y|\{x,y\}}$. Lastly, again by condition (γ), $H_{y|\{x,y\}} \geq_0 H^c_{y|B}$.

Case 2: $y \in A$, $x \notin B$
Note that $x \neq y$. As $A \neq X$, there exists some $z \in X\backslash A$, $z \neq x$, $z \neq y$. As $|X| \geq 4$, there exists some $z' \in X$ that is distinct from x, y, z. Applying conditions (γ) and (FNE), we have $H_{x|A} \geq_0 H^c_{x|\{x,z\}} \geq_0 H_{z|\{x,z\}} \geq_0 H^c_{z|\{z,z'\}} \geq_0 H_{z'|\{z,z'\}} \geq_0 H^c_{z'|\{x,z'\}} \geq_0 H_{x|\{x,z'\} \geq_0 H^c_{x|\{x,y\}}} \geq_0 H_{y|\{x,y\}} \geq_0 H^c_{y|B}$.

Case 3: $y \notin A$, $x \in B$
Note that $x \neq y$. As $B \neq X$, there exists some $z \in X\backslash B$, $z \neq x$, $z \neq y$. As $|X| \geq 4$, there exists some $z' \in X$ that is distinct from x, y, z. Applying conditions (γ) and (FNE), we have $H_{x|A} \geq_0 H^c_{x|\{x,y\}} \geq_0 H_{y|\{x,y\}} \geq_0 H^c_{y|\{y,z'\}} \geq_0 H_{z'|\{y,z'\}} \geq_0 H^c_{z'|\{z,z'\}} \geq H_{z|\{z,z'\}} \geq_0 H^c_{z|\{z,y\}} \geq_0 H_{y|\{y,z\}} \geq_0 H^c_{y|B}$.

Case 4: $y \in A$, $x \in B$
As $A, B \neq X$, there exist $z \in X\backslash A$, $z \neq x$, $z' \neq y$ and $z' \in X\backslash B$, $z' \neq x$, $z' \neq y$. If $z \neq z'$, we have by applying conditions (γ) and (FNE), $H_{x|A} \geq_0 H^c_{x|\{x,z\}} \geq_0 H_{z|\{x,z\}} \geq_0 H^c_{z|\{z,z'\}} \geq_0 H_{z'|\{z,z'\}} \geq_0 H^c_{z'|\{y,z'\}} \geq_0 H_{y|\{y,z'\}} \geq_0 H^c_{y|B}$. If $z = z'$, then there exists some z'' that is distinct from x, y, z. If $x \neq y$, then by applying conditions (γ) and (FNE) we have, $H_{x|A} \geq_0 H^c_{x|\{x,z\}} \geq_0$

$H_{z|\{x,z\}} \geq_0 H^c_{z|\{y,z\}} \geq_0 H_{y|\{y,z\}} \geq_0 H^c_{y|B}$. Else if $x = y$, there exists some z''' distinct from y, z, z''. Then, by applying conditions (γ) and (FNE), we have $H_{y|A} \geq_0 H^c_{y|\{y,z\}} \geq_0 H_{\{}z|\{y,z\}\} \geq_0 H^c_{z|\{z,z''\}} \geq_0 H_{z''|\{z,z''\}} \geq_0 H^c_{z''|\{z'',z'''\}} \geq_0 H_{z'''|\{z'',z'''\}} \geq_0 H^c_{z'''|\{z,z'''\}} \geq_0 H_{z|\{z,z'''\}} \geq_0 H^c_{z|\{y,z\}} \geq_0 H_{y|\{y,z\}} \geq_0 H^c_{y|B}$.

Consequently, in all cases, $H_{x|A} \geq H^c_{y|B}$. □

Proof of Theorem 1

Consider $\mathcal{D} = \mathcal{C}_{\text{bin}}$. Note that all $c \in \mathcal{D}$ satisfy (FNE), (α) and (γ). Let $x \in A \in \mathcal{A}_{\geq 2}$, $y \in B \in \mathcal{A}_{\geq 2}$. (i) Assume $x \neq y$ first. By Lemma 3, parts 1 and 2, $H^c_{x|A} \geq H_{y|B} \geq H^c_{y|B}$. By complementation-adaptedness, $H_{y|b} \geq H_{x|A}$. (ii) Now let $x = y$. Then pick any $z \neq x$ and some $C \in \mathcal{A}_{\geq 2}$ such that $z \in C$. By what was just shown, $H^c_{x|A} \geq H^c_{z|C} \geq H^c_{y|B}$. Again, by complementation-adaptedness, $H_{y|B} \geq H_{x|A}$. In total, we conclude that $H_{x|A} \equiv H_{y|B}$ and $H^c_{x|A} \equiv H_{y|B}$. Moreover, again by Lemma 3, parts 1 (and what we just showed for the case $x = y$) and 2, $H^c_{x|A} \geq H_{x|A} \geq H^c_{x|A}$; thus, $H^c_{x|A} \equiv H^c_{x|A}$. Consequently, $\mathcal{D} = \mathcal{C}_{\text{bin}}$ is totally blocked. Note that putting additional restrictions on choice correspondences introduces additional conditional entailments without removing existing one. Thus, all $\mathcal{D} = \mathcal{C}_{\text{qua}}, \mathcal{C}_{\text{wo}}, \mathcal{C}_{\text{lo}}$ are totally blocked. The results follows from (Nehring and Puppe, 2010, Theorem 1).

Proof of Theorem 2

Let $|X| \geq 4$ and suppose that all $c \in \mathcal{D}$ satisfy (FNE) and (γ). Let $A, B \in \mathcal{A}_{\geq 2}$ such that $A, B \neq X$. Consider any $x \in A$ and any $y \in B$. By Lemma 4, we have $H_{x|A} \geq H^c_{y|B}$. Let $z \in B$, $z \neq y$. Then, again by Lemma 4, $H_{x|A} \geq H^c_{z|B}$. By condition (FNE), $H^c_{z|B} \geq_0 H_{y|B}$. Thus, $H_{x|A} \geq H_{y|B}$. Moreover, letting $z' \in A$, $z' \neq x$ and using Lemma 4, condition (FNE) and what was just shown, we also have $H^c_{x|A} \geq_0 H_{z'|A} \geq H^c_{y|A}$ and $H^c_{x|A} \geq_0 H_{z'|A} \geq H_{y|A}$. Thus, $\mathcal{D}$ is totally blocked on all issues $\{H_{x|A}, H^c_{x|A}\}$ with $x \in A \neq X$. As there exists a critical family of length three on the collection of these issues (for example, for three distinct $x, y, z \in X$, consider $\{H_{x|\{x,y\}}, H_{x|\{x,z\}}, H^c_{x|\{x,y,z\}}\}$), some voter $j \in N$ has veto power (cf. 'Veto Lemma' in Nehring and Puppe, 2002/2010). As the subagenda of these issues is totally blocked, voter j is actually a dictator on it (cf. 'Contagion Lemma' in Nehring and Puppe, 2002/2010). Moreover, we have for all $A \neq X$ and all x, $H_{x|A} \geq_0 H_{x|X}$ (and thus, by complementation-adaptedness, $H^c_{x|X} \geq_0 H^c_{x|A}$). Thus, j can veto against society not choosing alternative $x \in X$. This means that we must have $f(c_1, \ldots, c_n)(X) \supseteq c_j(X)$.

Proof of Theorem 3

Consider $\mathcal{D} = \mathcal{C}_{\text{psd}}$. Note that all $c \in \mathcal{D}$ satisfy (FNE), (α) and (AIZ). Let $x \in A \in \mathcal{A}_{\geq 2}$, $y \in B \in \mathcal{A}_{\geq 2}$. By Lemma 3, part 3, $H_{x|A} \equiv H_{y|B}$ and $H^c_{x|A} \equiv H^c_{y|B}$. Moreover, letting $z \neq x$ and $z \in C \in \mathcal{A}_{\geq 2}$ and using the result just obtained together with part 1 of Lemma 3, we have $H^c_{x|A} \geq H_{z|C} \geq H_{y|B}$. At the same time, seeing that all critical families contain at most one un-negated property, $H_{x|A} \not\geq H^c_{y|B}$. Thus, $\mathcal{D} = \mathcal{C}_{\text{psd}}$ is semi-blocked. By Nehring (2006), an Arrovian aggregation rule is consistent on $\mathcal{C}_{\text{psd}}$ if and only if it an *oligarchy*. That is there exists some $M \subseteq \{1, \ldots, n\}$ such that for $(c_1, \ldots, c_n) \in \mathcal{C}^n_{\text{psd}}$ and all $A \in \mathcal{A}_{\geq 2}$, all $x \in A$: $x \in f(c_1, \ldots, c_n)(A) \iff \exists i \in M : x \in c_i(A)$. Thus, $f(c_1, \ldots, c_n)(A) = \bigcup_{i \in M} c_i(A)$.

Proof of Theorem 4

The equivalence of anonymous Arrovian aggregation to the existence of quotas $q_{x \in c(A)}$ is established in (Nehring and Puppe, 2010, Proposition 2.2).[19] Menu-level neutrality requires that, for some given menu $A \in \mathcal{A}_{\geq 2}$, all 'local' aggregation rules applied to the issues $x \in c(A)$ are identical. That is, the structure of winning coalitions must be the same for all such issues.

Now consider any $A \in \mathcal{A}_{\geq 2}$.

(i) Suppose $n \leq |A|$. If, for all $x \in A$, $0 < q_{x \in c(A)} \leq 1/n$, then each single individual forms a winning coalition for all $x \in c(A)$. As all individual choice sets $c_i(A)$ are non-empty, so is the collective choice set $c(A)$. On the other hand, if $q_{x \in c(A)} > 1/n$ for some $x \in A$, then $q_{x \in c(A)} > 1/n$ for all $x \in A$ (as all winning coalitions need to be the same). Consider some $(c_1, \ldots, c_n) \in \mathcal{C}^n_{\text{fne}}$ such that each individual chooses one (and only one) distinct alternative from A. Then $c(A) = \emptyset$.

(ii) Suppose $n > |A|$ and let $r = n \mod |A|$. Suppose that $0 < q_{x \in c(A)} \leq \frac{1}{|A|}(1 - \frac{r}{n}) + \frac{1}{n}\mathbb{1}(r \neq 0)$ for all $x \in A$. All individual choice sets $c_i(A)$ are non-empty. Thus, if $r = 0$, there exists some alternative $x \in A$ such that at least fraction $1/|A|$ of all individuals choose x from A. If $r > 0$, there exists some alternative $x \in A$ such that at least fraction $\frac{1}{|A|}(1 - \frac{r}{n}) + \frac{1}{n}$ of all individuals choose x from A. Hence $x \in c(A)$ in both cases and $c(A) \neq \emptyset$. Conversely, suppose there exists some $x \in A$ such that

[19] Nehring and Puppe (2010) consider quotas q_H, q_{H^c} such that H resp. H^c are accepted iff the fraction of voters supporting H resp. H^c is *strictly* greater than q_H and q_{H^c} respectively to treat both H and H^c equally. We only demand that support in favor of accepting x as collectively choosable from A *weakly* exceeds the quota $q_{x \in c(A)}$. On the other hand, the fraction of voters not choosing x from A needs to *strictly* exceed $q_{x \notin c(A)} := 1 - q_{x \in c(A)}$ to ban x from $c(A)$. However, except for very special cases (e.g., when $q_{x \in c(A)} = 1/2$ and n is even), these formulations are equivalent.

$q_{x\in c(A)} > \frac{1}{|A|}(1-\frac{r}{n}) + \frac{1}{n}\mathbb{1}(r \neq 0)$. Let $x_1, \ldots, x_k \in X$ be such that $A = \{x_1, \ldots, x_k\}$. Consider a profile $(c_1, \ldots, c_n) \in \mathcal{C}^n_{\text{fne}}$ such that $c_i(A) = x_j$, $j = 1, \ldots, n-r$ for $(n-r)/|A|$ individuals each and $c_i(A) = x_j$, $j = n-r+1, \ldots, n$ for $(n-r)/|A|+1$ individuals each (note that the total sum of individuals is thus n). Thus, the fraction of individuals voting for $x \in c(A)$ is less than $\frac{1}{|A|}(1-\frac{r}{n}) + \frac{1}{n}\mathbb{1}(r \neq 0)$ for each $x \in A$. As all structures of winning coalition are the same, this implies that $c(A) = \emptyset$.

B. Proof for Section 4

The proof of Proposition 1 is given in the main text.

Working Paper Series in Economics

recent issues

No. 173 *Claudio Kretz, Clemens Puppe:* Arrovian independence and the aggregation of choice functions, July 2025

No. 172 *Clemens Puppe, Attila Tasnadi:* Aggregating incomlete lists, July 2025

No. 171 *Clemens Puppe, Attila Tasnadi:* Violin virtuosi: Do their performances fade over time?, July 2025

No. 170 *Andranik S. Tangian:* Analysis of the 2025 Bundestag elections. Part 4 of 4: Changes in the German political spectrum, June 2025

No. 169 *Andranik S. Tangian:* Analysis of the 2025 Bundestag elections. Part 3 of 4: The third vote perspective, June 2025

No. 168 *Andranik S. Tangian:* Analysis of the 2025 Bundestag elections. Part 2 of 4: Representativeness of the parties and the Bundestag, June 2025

No. 167 *Andranik S. Tangian:* Analysis of the 2025 Bundestag elections. Part 1 of 4: Imperfection of the electoral reform, June 2025

No. 166 *Jessica Jachimowicz, Clemens Puppe:* Deliberation in group decisions: Polarization and Like-mindedness, May 2025

No. 165 *Kay Mitusch:* Aktuelle Entwicklungen im Eisenbahnsektor: Wettbewerb, verkehrspolitische Zielsetzungen, Steuerung der DB InfraGO und Trassenpreise, April 2025

No. 164 *Matthias Ammann, Clemens Puppe:* Preference diversity, April 2024

No. 163 *Andreas Benz, Peter R. Demerjian, Daniel Hoang, Martin Ruckes:* Picking winners: Managerial ability and capital allocation, January 2024

The responsibility for the contents of the working papers rests with the author, not the Institute. Since working papers are of a preliminary nature, it may be useful to contact the author of a particular working paper about results or caveats before referring to, or quoting, a paper. Any comments on working papers should be sent directly to the author.