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by Cyril Rouault and Resul Zoroglu

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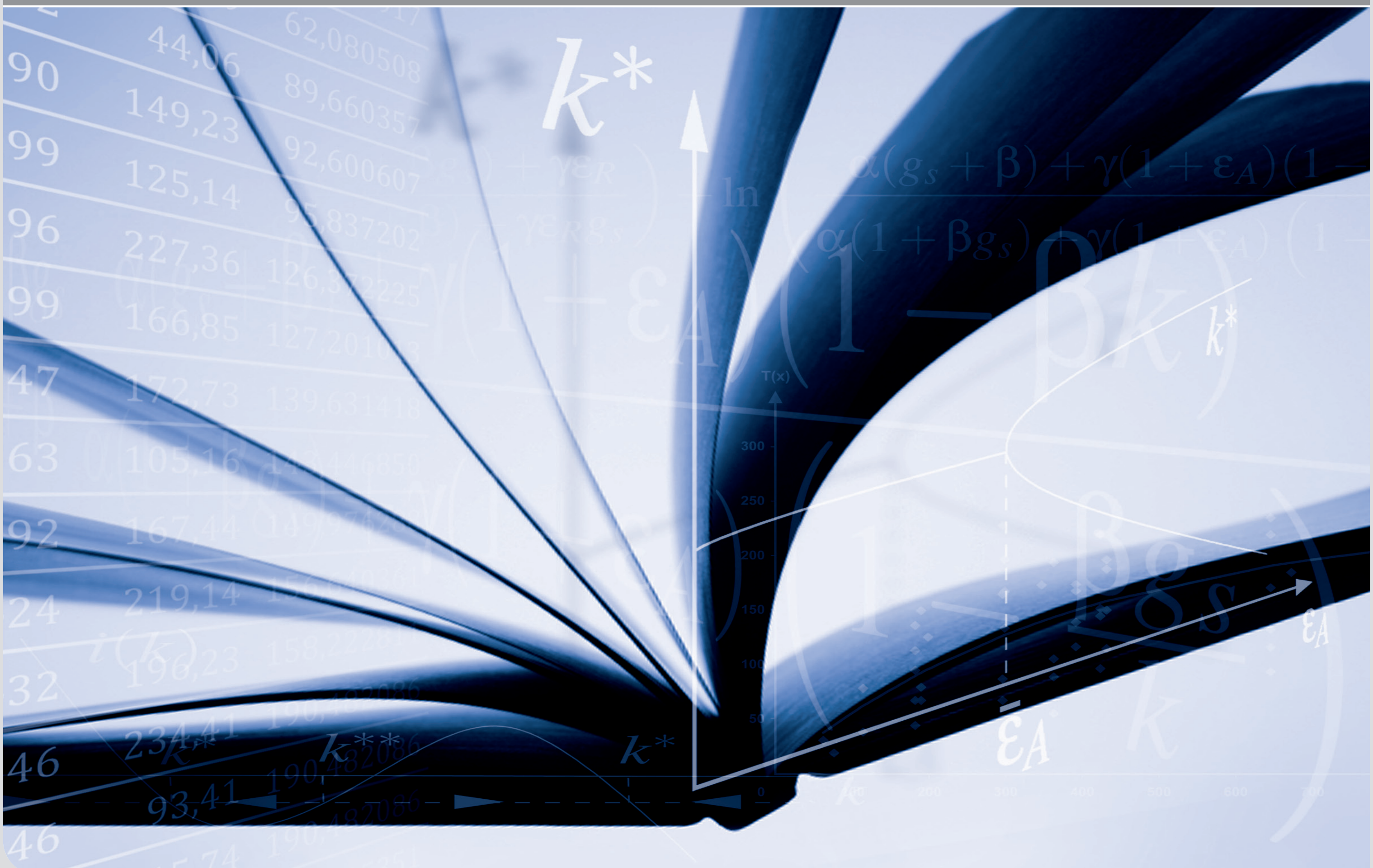

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Designing Rebate Rules in Public Goods Provision: Axioms, Limits, and Comparisons*

Cyril Rouault [†] Resul Zoroğlu [§]

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Abstract

This paper examines rebate rules in the context of public goods provision. These rules aim to redistribute the surplus when total contributions exceed the cost of the project. Using an axiomatic approach, we establish impossibility results that highlight the inherent tensions between fairness, participation incentives, and contribution incentives. We then propose and characterize the *Proportional Rebate with Threshold* rule, which identifies a coherent trade-off across these objectives.

Keywords: Public goods provision, Crowdfunding, Axioms, Rebates, Fairness.

JEL Classification: D63, D71, D82, G32.

1 Introduction

Public goods provision often involves collective financing arrangements in which individuals, firms, or municipalities pool resources to support a project. A well-known example is *crowdfunding*, which has become a prominent mechanism in contexts ranging from innovation (Miglo, 2022) and investment (Strausz, 2017) to public goods provision itself (Spencer et al., 2009). A defining feature of many such mechanisms is that, if total contributions fall short of the required amount, they are reimbursed to contributors.[1] However, when contributions exceed the project's cost in *provision point* mechanisms, the surplus is typically retained, which may discourage participation if contributors fear overcontributing. This paper develops an axiomatic approach to rebate rules that balance contribution incentives and mitigate these concerns.

In provision point mechanisms—including many crowdfunding campaigns—the process typically operates in two stages. In the first stage, potential contributors are informed about the project and its cost, which determines the minimum amount required for implementation.

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[†]Université Paris-Saclay, ENS Paris-Saclay, CEPS, cyril.rouault@universite-paris-saclay.fr

[‡]Karlsruhe Institute of Technology (KIT), resul.zoroglu@kit.edu

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[1]In the public goods literature, this is often referred to as *reimbursement* or a *money-back guarantee* (see Marks and Croson, 1998; Zubrickas, 2014).

Based on this information, they decide whether to contribute. The project is implemented if total contributions meet or exceed this cost by the end of the campaign. Otherwise, contributions are returned to participants, as in the *all-or-nothing* mechanism, or retained, as in the *keep-it-all* mechanism.[2] A central challenge in this stage is the *assurance problem*, which occurs when contributors hesitate to commit funds because they lack confidence that the project will reach its funding target (Isaac et al., 1989). In the second stage, any *surplus*—that is, contributions beyond the project's cost—may be redistributed among contributors. The treatment of this surplus has important implications for contribution decisions (Spencer et al., 2009), yet the literature has focused mainly on the first stage, leaving the second stage relatively unexplored. In this paper, we focus exclusively on the rebate of surplus contributions, excluding any distribution related to the project's completion.

A fundamental objective of public goods provision is to reach the funding requirement needed to implement the project. Achieving this depends on two key factors: the number of contributors and the size of their individual contributions. Both are essential to ensure the project's feasibility—broad participation increases the likelihood of reaching the cost, while higher individual contributions raise the total funds available. These objectives and factors are similar in crowdfunding campaigns, where success likewise hinges on attracting many backers and encouraging generous pledges. In both contexts, the design of rebate rules should support these goals by fostering participation and incentivizing higher contributions.

We adopt a model where individuals contribute a predetermined, non-strategic amount, and the total contributions determine the overall project cost. The project itself comprises multiple alternatives, each corresponding to a different level of funding. Depending on the total funds raised, the highest feasible alternative is selected, which naturally includes all lower-cost alternatives as part of the project's development. This framework mirrors real-world cases where a project's scale or quality expands with additional funding—such as upgrading a public facility with new features or progressing through successive stages of a development plan. This selection rule is consistent with the *Utilization Rebate* approach adopted by Marks and Croson (1998), who show that allocating surplus to finance a more advanced alternative is generally preferred by contributors and can enhance welfare. Once the total amount raised is determined and the project's cost is set, we analyze how any surplus portion of the contributions is redistributed. To structure this analysis, we define three groups of axioms. The first establishes a fundamental fairness principle for the rebate of surplus. The following two capture the main objectives of crowdfunding: incentivizing contributions and maximizing participation.

These groups are formalized as follows. Fairness is imposed by ensuring that contributors who *contribute equally receive equal rebate.* To incentivize contributions, we introduce a *monotonicity* axiom, which establishes an economic relationship whereby, under a higher contribution alternative by an individual, their expected rebate is correspondingly higher. This condition serves to mitigate strategic behaviors aimed at avoiding overcontribution. Finally, to encourage widespread participation, we impose that adding contributors should not drastically

[2] The all-or-nothing mechanism is used by platforms such as Kickstarter, whereas Indiegogo employs the keep-it-all mechanism. See Coats et al. (2009), Chemla and Tinn (2020) and Cumming et al. (2020) for further discussion.

alter the rebate received by individuals. The *population monotonicity* axiom accommodates projects spanning multiple municipalities or successive rounds of calls for contributions.

While combining these three axiomatic principles is desirable in rebate rule design, our results suggest that satisfying all three simultaneously is not possible. Specifically, we demonstrate that no rebate rule can simultaneously ensure equal treatment and incentivize higher individual contributions (Theorem 1). Moreover, the only rebate rule that guarantees equal treatment while encouraging an expansion of the contributing population is one that refrains from redistributing any surplus at all (Theorem 2).

We introduce weaker versions of these axioms to address these limitations and identify a rebate rule that better balances these objectives. Our proposal, the *Proportional Rebate with Threshold* (PRT) rule, redistributes the surplus only to contributors whose individual contributions exceed the *project's average cost*—referred to here as the *threshold*. For these contributors, the rebate reduces their *net payment*—defined as their contribution minus the rebate—toward that cost. By construction, the PRT rule directs a larger share of the surplus toward higher contributors, even when contributions—and, in practice, valuations—differ across individuals. We characterize the PRT rule and show that it is the only rule that strikes a balance among the desirable axioms, simultaneously promoting high contributions, broad participation, and fairness.

We introduce the *Marginal Rebate of Contribution* (MRC) as a tool to assess how an individual's rebate changes in response to a hypothetical increase in their contribution, while holding others' contributions constant. Although contributors in our model do not behave strategically, analyzing the MRC allows us to explore the potential incentives that could influence contribution behavior in alternative scenarios. This conceptual exercise helps illuminate how rebate rules might encourage or discourage higher contributions. We demonstrate that the PRT rule sustains a higher rebate in cases of over-contribution compared to rebate rules used in fair-division settings, such as the *Shapley* rule and the widely used *Proportional Rebate* rule. More precisely, under these latter rules, when individuals contribute at a higher level, a significant portion of their excess contribution is redistributed to other contributors. We argue that the PRT rule mitigates this effect, making it a more effective rule for balancing incentives and fairness.

Related Literature

An extensive literature on public goods provision examines mechanisms that encourage individual contributions while mitigating *free-riding*, whereby individuals withhold contributions in the expectation that others will fund the project. Similarly, crowdfunding mechanisms often incorporate rewards to motivate contributors—these can take various forms, such as discounted rates, exclusive products, or equity stakes in the funded venture. Moreover, the closely related assurance problem plays a central role in both contexts; Isaac et al. (1989) argue that it is even more critical than free-riding. A common solution is the *money-back guarantee*, which ensures contributors are reimbursed if the project fails to reach its funding target. In contrast, the keep-it-all mechanism allows campaign initiators to retain all contributions regardless of funding success, potentially undermining contributors' confidence and reducing participation. Experimental and empirical evidence (e.g., Isaac et al., 1989; Rondeau et al., 1999; Coats et al., 2009; Chemla and Tinn, 2020; Cumming et al., 2020) consistently shows that money-back guarantees increase contributions and improve project completion rates. Moreover, Bagnoli and Lipman (1989) establish that such guarantees fully implement the core within voluntary contribution mechanisms, thereby promoting efficient public goods provision.

Rebate rules offer an alternative approach to enhancing contribution incentives in the post-funding stage. Spencer et al. (2009) compare six different rebate rules and find that the Proportional Rebate is the most effective in achieving funding targets.[3] However, their analysis does not include the PRT rule and remains silent on its relative effectiveness. More recently, Oezcelik et al. (2025) introduce the *Bid-Cap* rule, which restricts the highest individual contributions to prevent excessive payments once the funding goal is met. In our model, the set of alternatives includes a null project with zero cost. Within this framework, the PRT rule satisfies a money-back guarantee: contributors are fully reimbursed whenever no project is implemented. In addition, we formalize contribution incentives through the axioms of *Contribution Monotonicity* and *Monotonicity of Net Payment*, which together ensure that (i) when an individual revises her contribution and chooses to increase it, her rebate rises, and and (ii) across individuals, a higher contribution leads to a weakly higher rebate, without ever resulting in a lower net payment for the higher contributor. To compare the extent to which different rules reward higher contributions, we rely on the MRC, which we interpret as the incremental change in an individual's rebate when considering alternative (higher) contributions. We show that the MRC under the PRT rule exceeds that obtained under two natural benchmark rules, namely the Proportional Rebate and the Shapley rules.[4] This suggests that, when rebates are used, the PRT rule provides stronger incentives for contributors to increase their contributions than these alternative rules. To the best of our knowledge, this paper is the first to use the MRC as a criterion for comparing rebate rules.

Another approach to rebate mechanisms involves using surplus funds to expand or enhance the project itself. This concept is formalized in the Utilization Rebate rule, introduced by Marks and Croson (1998). A classic example is found in projects characterized by continuously variable funding levels—where additional contributions can be proportionally allocated—such as tree-planting initiatives, where surplus contributions beyond the initial project cost are allocated to additional planting efforts. However, many projects—particularly public facilities such as municipal swimming pools, parks, or daycare centers—experience improvements in discrete stages, corresponding to specific levels of funding rather than a continuous scale. To accommodate this, we model the project as consisting of several distinct alternatives, each corresponding to an enhanced version of the project with a higher funding level. This framework reflects the fact that additional financial resources enable discrete expansions or

[3]Our framework differs from the classical surplus-sharing model of Moulin (1987). In Moulin's setting, surplus arises from the efficient allocation of a fixed bundle under quasi-linear utilities. In contrast, in provision-point mechanisms such as ours, surplus is generated only *ex post*, when aggregate contributions exceed the project's cost.

[4]Although the Shapley value is widely used in cost-sharing, fair division, and reward-allocation settings (Shapley, 1953; Moulin and Shenker, 1992), it is typically defined on total costs or total values. The Shapley rule introduced in Section 5 instead applies the Shapley value to the surplus arising in a provision-point mechanism, thereby generating new insights for public-good provision and crowdfunding.

upgrades, implemented in stages rather than on a continuous scale. Moreover, when the project is unique and no alternative exists, the Utilization Rebate rule remains silent, whereas the PRT rule still provides a rebate.

Our paper also contributes to the axiomatic literature on public goods provision. Béal et al. (2025) highlight the role of early backer incentives and employ an axiomatic approach to characterize reward mechanisms that encourage early contributions. However, this line of research primarily focuses on pre-funding incentives and overlooks the management of surplus funds once the provision point has been reached. Our paper extends this literature by providing an axiomatic analysis of post-funding surplus rebates and suggesting that the rebate amount can be an important factor influencing participation incentives.

The structure of the paper is as follows: Section 2 introduces the model and presents axioms used in our analysis. Section 3 establishes fundamental limitations in the design of rebate rules. In Section 4, we introduce the Proportional Rebate with Threshold rule and examine the axioms it satisfies. Section 5 provides a comparative analysis of rebate rules. Section 6 concludes with key insights and directions for future research. Appendix A provides an additional insight, and all proofs are collected in Appendix B.

2 Model

This section introduces our model and the axioms we consider.

2.1 Rebate Problem

Let $I = \{i_1, ..., i_n\}$ be a set of *individuals*, and $P = \{P_0, P_1, ..., P_m\}$ be an ordered set of *project alternatives*, where P_0 represents the *null alternative* (no project is funded). Each alternative P_k represents a progressively expanded version of the project, with associated costs $c_k \in \mathbb{R}_+$ such that for any $k, k' \in \{0, ..., m\}$, with $k < k'$, we have $c_k < c_{k'}$. The cost of the null alternative P_0 is normalized to $c_0 = 0$. We assume that $m \geq 1$, meaning that at least one non-null project alternative exists. In our approach, we consider the *average cost* of a funded project alternative P_k, denoted by $\bar{c}_k \equiv \frac{c_k}{|I|}$.

Each agent $i \in I$ *contributes* a non-negative amount $x_i \in \mathbb{R}_+$. Let $\mathbf{x} \equiv (x_i)_{i\in I}$ be the *contribution vector*. The *total amount collected* is given by $X \equiv \sum_{i\in I} x_i$. A project alternative P_k is *funded* if and only if $c_k \leq X < c_{k+1}$.[5] The *surplus* $S \equiv X - c_k$, represents the portion of X not used to finance the project. A (*rebate*) *problem* is defined by the tuple $(I, P, \mathbf{x})$. Let Π be the *set of all problems*.

Given a problem $(I, P, \mathbf{x}) \in \Pi$, a *rebate* is a vector of positive real numbers $\mathbf{r} \equiv (r_i)_{i\in I} \in \mathbb{R}_+^{|I|}$ such that $\sum_{i\in I} r_i \leq S$. Let $R(I, P, \mathbf{x})$ denote the set of all rebates for the problem $(I, P, \mathbf{x})$. A (*rebate*) *rule* is a mapping $\varphi : \Pi \to \mathbb{R}_+^{|I|}$ that assigns, to each problem $(I, P, \mathbf{x}) \in \Pi$, a

[5] We assume that the funded alternative is the most expensive one among those whose cost does not exceed the total amount collected. This aligns with the logic of the Utilization Rebate rule (see Marks and Croson, 1998), whereby using rebates to support a larger-scale project effectively functions as a preferred form of rebate compared to direct monetary returns. Implicitly, we assume that projects are comparable in nature and improve only in quality as the amount collected increases. If the implemented project were to change fundamentally with the total amount collected, contributors' incentives would be affected, as individuals may hold preferences over specific projects rather than solely over project scale or quality.

rebate $\varphi(I, P, \mathbf{x}) \in R(I, P, \mathbf{x})$. Given rule φ and $(I, P, \mathbf{x})$, we denote by $\varphi_i(I, P, \mathbf{x})$ the rebate of individual i.

2.2 Axioms

We now consider five axioms for rebate rules. The first axiom reflects a standard fairness principle. It states that if two agents contribute the same amount, they must receive identical rebates.

Equal Treatment of Equals (ETE). A rule rule φ satisfies (ETE) if for each problem $(I, P, \mathbf{x}) \in \Pi$, for all individuals $i, j \in I$ such that $x_i = x_j$, we have $\varphi_i(I, P, \mathbf{x}) = \varphi_j(I, P, \mathbf{x})$.

The second axiom guarantees that a contributor is not put at a disadvantage when comparing different possible contribution levels. Specifically, if an individual's contribution in one scenario is higher than in another, then either a more expensive project alternative is funded, or their rebate amount increases by at least the difference in contributions.[6]

Contribution Monotonicity (CM). A rule φ satisfies (CM) if, for each problem $(I, P, \mathbf{x}) \in \Pi$, with P_k being the funded alternative, the following holds:

For each $i \in I$ such that $x_i \geq \bar{c}_k$,[7] consider an alternative contribution $x'_i > x_i$. Then either

$$\varphi_i(I, P, (\mathbf{x}_{-i}, x'_i)) - \varphi_i(I, P, \mathbf{x}) \geq x'_i - x_i,$$ [8]

or

$$x'_i + \sum_{j \in I \setminus \{i\}} x_j \geq c_{k'} \text{ with } k' > k.$$

The third axiom relies on the same reasoning but considers an alternative scenario in which an individual increases her contribution. It requires that such an increase should not reduce the rebate received by any other individual, or that it leads to the funding of a more costly project alternative.

Global Monotonicity (GM). A rule φ satisfies (GM) if, for each problem $(I, P, \mathbf{x}) \in \Pi$, with P_k being the funded alternative, for $i \in I$ with $x'_i > x_i$, it holds for each $j \in I$ that

$$\varphi_j(I, P, (\mathbf{x}_{-i}, x'_i)) \geq \varphi_j(I, P, (\mathbf{x}),$$

or

$$x'_i + \sum_{j \in I \setminus \{i\}} x_j \geq c_{k'} \text{ with } k' > k.$$

The fourth axiom ensures that if an individual benefits from rebate, then she continues to receive a positive rebate when new contributors join the population.

[6]Bagnoli and Lipman (1989) suggest that rebate rules should have the property that an increase in the contribution of \$1 by individual i should not generate a rebate to individual i of more than \$1. We formalize this reasoning in more detail in Section 5.

[7]We rely on the project's average cost as a benchmark to capture sufficiently high individual contributions, while imposing no constraint when contributions are below this threshold.

[8]Note that this inequality can equivalently be written as

$$\varphi_i(I, P, (\mathbf{x}_{-i}, x'_i)) - x'_i \;\geq\; \varphi_i(I, P, \mathbf{x}) - x_i,$$

which makes explicit the monotonicity of the individual's net payment.

Strong-Population Monotonicity (Strong-PM). A rule φ satisfies (Strong-PM) if, for each problem $(I, P, \mathbf{x}) \in \Pi$ and for any partition $\mathcal{P} = \{I', I''\}$ of I such that $I' \cup I'' = I$ and $I' \cap I'' = \emptyset$:

$$\forall i \in I', \varphi_i(I', P, \mathbf{x}') > 0 \Rightarrow \varphi_i(I, P, \mathbf{x}) > 0, \text{ and } \forall i \in I'', \varphi_i(I'', P, \mathbf{x}'') > 0 \Rightarrow \varphi_i(I, P, \mathbf{x}) > 0,$$

where $\mathbf{x}' \equiv (x_i)_{i \in I'}$ and $\mathbf{x}'' \equiv (x_i)_{i \in I''}$.
This ensures that an agent's rebate does not become null solely due to the addition of new contributors. As discussed in the introduction, this axiom is crucial for encouraging broad participation. No restriction is imposed on the partition of individuals, thereby accommodating practical situations involving multiple municipalities or successive rounds of contribution calls.
The next axiom ensures that whenever there is a positive surplus, at least part of it must be redistributed.
Partial Surplus Rebate (PSR). A rule φ satisfies (PSR) if, for each problem $(I, P, \mathbf{x}) \in \Pi$, $S > 0$ implies that there exists $i \in I$ such that $\varphi_i(I, P, \mathbf{x}) > 0$.
We also adopt a stronger requirement ensuring that the entire surplus is redistributed.
Full Surplus Rebate (FSR). A rule φ satisfies (FSR) if, for each problem $(I, P, \mathbf{x}) \in \Pi$, $\sum_{i \in I} \varphi_i(I, P, \mathbf{x}) = S$.

3 Limits on Rebate Rules

In this section, we highlight the main limitations regarding the compatibility of the axioms presented in the previous section with rebate rules. Our first result establishes a fundamental limitation regarding contribution inducement and equal treatment.

Theorem 1. There is no rule φ that satisfies Equal Treatment of Equals (ETE) and Contribution Monotonicity (CM).

Proof. See Appendix B.1. ■

The implication of Theorem 1 is that ensuring equal treatment requires redistributing part of any alternative contribution level beyond the initial one to others. In practice, this weakens individual incentives to contribute, as they anticipate that part of any additional amount they give will neither be returned to them nor directly benefit the project.
As discussed in the introduction, rebate is absent in many crowdfunding mechanisms. The following definition formalizes the *Null Rebate* rule.

Definition 1. (Null Rebate rule). The Null Rebate rule φ^0 is defined for each $(I, P, \mathbf{x}) \in \Pi$ as:

$$\forall i \in I, \varphi_i^0(I, P, \mathbf{x}) = 0.$$

In other words, φ^0 specifies that the surplus is not redistributed to the participants. Our second result establishes a limitation on surplus rebate when (ETE) and (Strong-MM) are required.

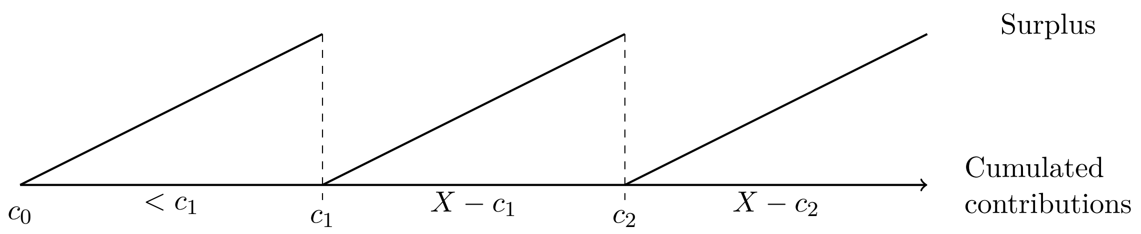

Figure 1: Illustration of Available Surplus

Theorem 2. A rule φ satisfies Equal treatment of equals (ETE) and Strong-Population Monotonicity (Strong-PM) if, and only if, $\varphi = \varphi^0$.

Proof. See Appendix B.2. ■

Theorem 2 is revealing, as it shows that designing a rule that satisfies both (ETE), a fundamental fairness requirement, and (Strong-MM), which preserves the existence of positive rebates when new contributors are added, is only possible when no surplus is redistributed. In the context of crowdfunding, Theorem 2 implies that if a positive rebate is desired by adding contributors, it is not possible to treat them equally. Our next result further extends this limitation.

Theorem 3. There is no rule φ that satisfies Strong-Population Monotonicity (Strong-PM) and Partial Surplus Rebate (PSR).

Proof. See Appendix B.3. ■

The intuition behind this impossibility stems from the nature of alternative funding. When contributions are added, a project (or a more expensive alternative) is funded, which reduces the available surplus. As a result, it becomes impossible to have a partial rebate for individuals, and the addition of new individuals leads to a zero surplus. Figure 1 illustrates this phenomenon.

4 Proportional Rebate with Threshold Rule and Axioms

In this section, we introduce a rebate rule designed to address the limitations discussed earlier, while aiming to satisfy key axioms essential to provision point mechanisms. Beyond the axioms presented, rebate rules implemented in practice typically satisfy two additional conditions. First, if no project is funded, a *money-back guarantee* returns contributions to participants.[9] Second, if there is no surplus, no rebate occurs. The *Proportional Rebate with Threshold* (PRT) rule satisfies both of these conditions. It operates based on the project's average cost, ensuring that rebates adjust contributors' *net payment*—defined as their contribution minus the rebate—toward that average. Before defining the PRT rule, we introduce two new notations. Let $\hat{X} \equiv \sum_{j \in \{j \in I : x_j > \bar{c}_{k^*}\}} (x_j - \bar{c}_{k^*})$, and $\check{X} \equiv \sum_{j \in \{j \in I : x_j \leq \bar{c}_{k^*}\}} (\bar{c}_{k^*} - x_j)$ where P_{k^*} is the project alternative funded.[10] Intuitively, $\hat{X}$ represents the total excess contributions

[9] Isaac et al. (1989) show that the money-back guarantee helps mitigate the assurance problem discussed in the Related Literature Section 1.

[10] Note that we have $S = \hat{X} - \check{X}$.

above the project's average cost, while $\check{X}$ captures the total shortfall of contributions below this threshold.

Definition 2. (Proportional Rebate with Threshold (PRT) rule). The rule ψ is defined, for each $(I, P, \mathbf{x}) \in \Pi$, with P_{k^*} being the funded project alternative, as for each $i \in I$,

$$\psi_i(I, P, \mathbf{x}) = \begin{cases} (x_i - \bar{c}_{k^*}) - \frac{x_i - \bar{c}_{k^*}}{\hat{X}} \times \check{X} & \text{if } x_i \geq \bar{c}_{k^*}, \\ 0 & \text{otherwise.} \end{cases}$$

The rule ψ rebates the surplus based on how each individual's contribution compares to the project's average cost. Participants who contribute more than the average cost receive a proportional rebate based on the excess contribution.[11] Those whose contributions are below the average cost do not receive any rebate. This rule adjusts the net payments according to each individual's contribution relative to the overall funding, while encouraging greater participation by offering a clear proportional rebate based on the project's success. We formally demonstrate this in the remainder of this section.

Given the impossibilities presented in Theorems 1 and 3, we propose weakened axioms to address these challenges. The first of these is a weakened form of population monotonicity. This axiom requires that when individuals are added to the population, the rebate either remains positive or allows for the financing of a project (or a more costly alternative).

Population Monotonicity (PM). A rule φ satisfies (PM) if, for each problem $(I, P, \mathbf{x}) \in \Pi$, and for any partition $\mathcal{P} = \{I', I''\}$ of I such that $I' \cup I'' = I$ and $I' \cap I'' = \emptyset$, with $\varphi(I', P, \mathbf{x}')$ with project alternative $P_{k'}$ funded, $\varphi(I'', P, \mathbf{x}'')$ with project alternative $P_{k''}$ funded, and $\varphi(I, P, \mathbf{x})$ with project alternative P_k funded, then the following conditions hold:

$$\forall i \in I', \varphi_i(I', P, \mathbf{x}') > 0 \Rightarrow \varphi_i(I, P, \mathbf{x}) > 0, \text{ or } k > k',$$

and,

$$\forall i \in I'', \varphi_i(I'', P, \mathbf{x}'') > 0 \Rightarrow \varphi_i(I, P, \mathbf{x}) > 0 \text{ or } k > k'',$$

where $\mathbf{x}' \equiv (x_i)_{i \in I'}$ and $\mathbf{x}'' \equiv (x_i)_{i \in I''}$.[12]

The second weakened axiom addresses the monotonicity of an individual's rebate in relation to their contribution. Specifically, it ensures that if an individual whose contribution exceeds the average cost of the funded project alternative opts for a higher contribution level, their rebate will increase by at least part of the additional amount, or a more costly project alternative will be funded. To formalize this, we consider the sum of the excess contributions from all individuals other than i, that is, $\hat{X}_{-i} \equiv \sum_{j \in \{j \in I \setminus \{i\}: x_j > \bar{c}_{k^*}\}} (x_j - \bar{c}_{k^*})$.

Weak-Contribution Monotonicity (Weak-CM). A rule φ satisfies (Weak-CM) if, for each problem $(I, P, \mathbf{x}) \in \Pi$, with P_k being the funded project alternative the following holds:

[11] We implicitly assume that at least one individual contributes more than the average cost so that $\hat{X} > 0$. By definition of the chosen funded project, we have $\hat{X} \geq 0$, and the case $\hat{X} = 0$ implies that all individuals contribute exactly $\bar{c}_{k^*}$. In this situation, there is no surplus to rebate, and thus a rebate rule becomes irrelevant. For this reason, the assumption is reasonable.

[12] Population Monotonicity is particularly relevant in the context of public good provision involving multiple municipalities. For instance, two neighboring towns aiming to build a public swimming facility can pool their contributions to fund a larger, improved alternative. Otherwise, any excess contributions are returned to them.

For each $i \in I$ such that $x_i \geq \bar{c}_k$, if the contribution of agent i increases to $x'_i > x_i$, then either

$$\varphi_i(I, P, (\mathbf{x}_{-i}, x'_i)) - \varphi_i(I, P, \mathbf{x}) \geq (x'_i - x_i) \times \left(1 - \frac{\check{X} \times \hat{X}_{-i}}{\left(\hat{X}_{-i} + x'_i - \bar{c}_{k^*}\right) \times \left(\hat{X}_{-i} + x_i - \bar{c}_{k^*}\right)}\right),$$

or

$$x'_i + \sum_{j \in I \setminus \{i\}} x_j \geq c_{k'} \text{ with } k' > k.$$

Note that the definition of (Weak-CM) uses the notations introduced in the PRT Rule. The underlying rationale is that the rebate must be a function of the individual's initial contribution, a higher alternative contribution they might select, and the contributions of other participants.

We now introduce the characterization of the PRT rule.

Theorem 4. A rule φ satisfies Equal Treatment of Equals (ETE), Full Surplus Rebate (FSR), Weak-Contribution Monotonicity (Weak-CM) and Global Monotonicity (GM) if, and only if, φ is the PRT rule ψ.

Proof. See Appendix B.4. ■

In the remainder of this section, we show that the PRT rule satisfies two additional desirable axioms. First, it satisfies Population Monotonicity, which promotes broader participation. Second, we introduce the axiom of *Monotonicity of Net Payment* (MNP), which requires that individuals who contribute more cannot end up with a lower net payment than those who contribute less. This condition preserves a coherent ordering of net payments across contributors and prevents manipulation incentives in which increasing one's contribution would yield a strictly more advantageous net outcome.

Monotonicity of Net Payment (MNP). A rule φ satisfies (MNP) if, for each problem $(I, P, \mathbf{x}) \in \Pi$, for each $i, j \in I$ such that $x_i \geq x_j$, then $x_i - \varphi_i(I, P, \mathbf{x}) \geq x_j - \varphi_j(I, P, \mathbf{x})$.

Remark 1. If a rule φ satisfies Monotonicity of Net Payment (MNP), then φ satisfies Equal Treatment of Equals (ETE).

The proof is immediate, setting $x_i = x_j$ in the definition of MNP yields $\varphi_i(I, P, \mathbf{x}) = \varphi_j(I, P, \mathbf{x})$.

Proposition 1. The PRT rule ψ satisfies Population Monotonicity (PM) and Monotonicity of Net Payment (MNP).

Proof. See Appendix B.5. ■

Theorem 4 and Proposition 1 establish that the rule ψ satisfies the central axioms governing fairness, incentives to contribute, and the ability to broaden participation.

The following remark provides further insight into the incentives to contribute when all participants contribute more than the average cost of the project, by redistributing the entire additional contribution.

Remark 2. Consider a problem $(I, P, \mathbf{x})$ such that P_k is funded. If for each $i \in I$, $x_i \geq \overline{c}_k$, then for each $i \in I$, $\psi_i(I, P, \mathbf{x}) = x_i - \overline{c}_k$.

Remark 2 indicates that when all contributions exceed the project's average cost, rebates equalize net payments across individuals: after rebates, each individual effectively contributes the same amount—namely, the average cost of the funded project.[13]

5 Comparison of Rebate Rules

In this section, we evaluate how different rules affect an individual's rebate when they increase their contribution. As shown in the proof of Theorem 4, an increase in an individual's contribution always leads to the same share of the rebate,[14] or the financing of a more costly project alternative. To quantify this effect, we define the *Marginal Rebate of Contribution* (MRC) as the rate of change in an individual's rebate with respect to their own contribution, while keeping others' contributions fixed.

Definition 3. (Marginal Rebate of Contribution). Given a rule φ, the Marginal Rebate of Contribution (MRC) of an individual i when their contribution changes from x_i to x_i' is defined as:

$$MRC(\varphi_i(I, P, \mathbf{x}), x_i') = \frac{\varphi_i(I, P, (\mathbf{x}_{-i}, x_i')) - \varphi_i(I, P, \mathbf{x})}{x_i' - x_i}.$$

The underlying argument is that the greater the increase in rebate in response to higher contributions, the more individuals will be incentivized to contribute.[15] We now introduce our two benchmark rules: the *Shapley* rule and the widely used *Proportional Rebate* (PR) rule.

Definition 4. (Shapley rule). The rule φ^S is defined for each $(I, P, \mathbf{x}) \in \Pi$, with P_{k^*} being the funded project alternative as:

$$\forall i \in I, \varphi_i^S(I, P, \mathbf{x}) = \sum_{T \subseteq I \setminus \{i\}} \frac{|T|!(|I| - |T| - 1)!}{|I|!} \left(v(T \cup \{i\}) - v(T)\right),$$

where $v(T) = \max\left\{0,\ \sum_{j \in T} x_j - c_{k^*}\right\}$ represents the surplus that coalition T would generate relative to the cost c_{k^*} of the funded project alternative.

The rule φ^S is based on the Shapley value, which is widely studied in the fair-division and cost-sharing literature. In our setting, the rule adapts the Shapley value to a surplus-redistribution problem in a provision point environment. In contrast to classical Shapley cost–sharing rules—typically defined in terms of project costs or total values—our formulation evaluates, for each coalition T, the surplus it generates relative to the cost c_{k^*} of the funded project alternative (Shapley, 1953).

[13]The proof is straightforward since $\check{X}$ is equal to 0, and is therefore omitted.

[14]This share corresponds to $\left(1 - \frac{\check{X} \times \hat{X}_{-i}}{(\hat{X}_{-i} + x_i' - \overline{c}_{k^*}) \times (\hat{X}_{-i} + x_i - \overline{c}_{k^*})}\right)$.

[15]Note that the MRC corresponds to the partial derivative of the rebate rule when the increase in individual i's contribution, namely $x_i' - x_i$, approaches 0.

Definition 5. (Proportional Rebate (PR) rule). The rule φ^P is defined for each $(I, P, \mathbf{x}) \in \Pi$, with P_{k^*} being the funded project alternative, as:

$$\forall i \in I, \varphi_i^P(I, P, \mathbf{x}) = \frac{x_i}{X} \times S.$$

Although the underlying argument behind φ^P and ψ is similar, the implementation of a threshold significantly influences the resulting rebates. In φ^P, the rebate is proportional for all individuals, whereas in ψ, only those whose contributions exceed the average cost are eligible for a rebate. Notably, this rebate is lower when the contributions of individuals increase.

Proposition 2. The Shapley rule φ^S and the Proportional Rebate rule φ^P do not satisfy Weak-Contribution Monotonicity (Weak-CM).

The consequence of Proposition 2 is that, when using rules φ^S and φ^P, a larger portion of the increase in individual contributions will be redistributed to others compared to rule ψ. To illustrate this phenomenon, consider the following examples.

Example 1. Consider a problem $(I, P, \mathbf{x})$ where $I = \{i_1, i_2, i_3\}$, $P = \{P_0, P_1\}$ with $c_1 = 12$, it follows that $\bar{c}_1 = 4$. Suppose $x_{i_1} = 7$, $x_{i_2} = 8$, $x_{i_3} = 4$ and $x'_{i_1} = 8$. Table 1 illustrates the rebates for each rule.

Rebates	**PRT**		**Shapley**		**PR**	
x_{i_1}	7	8	7	8	7	8
r_{i_1}	3	4	2.83	3.33	2.58	3.2
r_{i_2}	4	4	2.83	3.33	2.95	3.2
r_{i_3}	0	0	1.33	1.33	1.47	1.6
Fix values	$x_{i_2} = 8$		$x_{i_3} = 4$	$c_1 = 12$	$\bar{c}_1 = 4$	

Table 1: Comparison of rebates under different rules (PRT, Shapley, and PR) for Case 1 without compensatory amount (everyone contributes at least the average cost $\bar{c}_1 = 4$). The table presents the rebates $(r_{i_1}, r_{i_2}, r_{i_3})$ for different initial contributions x_{i_1} (7 and 8), given fixed values of $x_{i_2} = 8$, $x_{i_3} = 4$, total cost $c_1 = 12$, and average cost $\bar{c}_1 = 4$.

In this example, when the contribution of i_1 changes from 7 to 8, the MRC values for i_1 under each rule are:

- $MRC(\psi_i(I, P, \mathbf{x}), 8) = 1$,
- $MRC(\varphi_i^S(I, P, \mathbf{x}), 8) = 0.50$, and
- $MRC(\varphi_i^P(I, P, \mathbf{x}), 8) = 0.62$.

This example is detailed in Figure 2, where the variation in i_1's contribution corresponds to Case 1. In this case, the contributions of i_2 and i_3 are sufficient to fund the project, meaning there is no compensatory amount. The following table further demonstrates that this observation holds even when accounting for the compensatory amount, i.e., $\check{X} > 0$.

We consider another example where the contributions of i_2 and i_3 differ, with $x_{i_2} = 7$ and $x_{i_3} = 2$. Table 2 shows that when i_1's contribution changes from 7 to 8, the MRC values for i_1 under each rule are:

Rebates	PRT		Shapley		PR	
x_{i_1}	7	8	7	8	7	8
r_{i_1}	2	2.86	1.67	2.17	1.75	2.35
r_{i_2}	2	2.14	1.67	2.17	1.75	2.06
r_{i_3}	0	0	0.67	0.67	0.50	0.59
Fix values	$x_{i_2} = 7$		$x_{i_3} = 2$	$c_1 = 12$	$\bar{c}_1 = 4$	

Table 2: Comparison of rebates under different rules (PRT, Shapley, and PR) for Case 2 with compensatory amount (someone contributes less than the average cost $\bar{c}_1 = 4$). The table presents the rebates $(r_{i_1}, r_{i_2}, r_{i_3})$ for different initial contributions x_{i_1} (7 and 8), given fixed values of $x_{i_2} = 7$, $x_{i_3} = 2$, total cost $c_1 = 12$, and average cost $\bar{c}_1 = 4$.

- $MRC(\psi_i(I, P, \mathbf{x}), 8) = 0.86$,
- $MRC(\varphi_i^S(I, P, \mathbf{x}), 8) = 0.50$, and
- $MRC(\varphi_i^P(I, P, \mathbf{x}), 8) = 0.60$.

The reasoning behind this observation is that a larger portion of i_1's additional contribution is redistributed to the other individuals. Specifically, the amount redistributed to i_2 is due to i_3's contribution being lower than $\bar{c}_1$. These values correspond to Case 2 in Figure 2.

Figure 2 illustrates the rebates of the three rules considered in this section across five distinct cases. These cases vary individual contributions while keeping the total cost $c_1 = 12$ and the average cost $\bar{c}_1 = 4$ constant. The key distinction among them lies in how individual contributions compare to the average cost. By systematically adjusting this relationship, each scenario highlights a fundamental aspect of the rebate rules. Additional cardinal combinations would merely interpolate among these representative cases, adding minimal incremental insight, as the essential characteristics of rebates have already been captured.

A general observation is that the PRT rule yields the highest MRC values when the initial contribution exceeds the average cost threshold $\bar{c}_1$, prioritizing over-contributors in the surplus allocation. For clarity, the direct comparison of the cases is presented in Figure 3.

Case 1 presents a scenario in which one contributor provides a higher contribution ($x_{i_2} = 8 > \bar{c}_1$) while the other contributes exactly at the threshold ($x_{i_3} = 4 = \bar{c}_1$), with their combined contribution exactly covering the project cost ($x_{i_2} + x_{i_3} = 12 = c_1$).

In this case, the PRT rule differs markedly from the other rules. The rebate of i_2 increases as x_{i_1} rises, absorbing part of i_1's contribution until i_2's net payment equals the project's average cost $\bar{c}_1$. Similarly, i_1 begins to receive a rebate once $x_{i_1} = 4 = \bar{c}_1$, capturing the entire surplus generated beyond this point. Under the other rules, all individuals start receiving a rebate as soon as the surplus becomes positive. As a result, the rebates of i_1 and i_2 are necessarily lower, since i_3 absorbs part of the surplus.

Case 2 involves one contribution below $\bar{c}_1$ ($x_{i_3} = 2 < \bar{c}_1$) while the other exceeds it ($x_{i_2} = 7 > \bar{c}_1$). Yet the total contribution is insufficient to cover the cost ($x_{i_2} + x_{i_3} = 9 < c_1$). For the PRT rule, as long as x_{i_1} remains below $\bar{c}_1$, $\psi_{i_2}(I, P, \mathbf{x})$ increases linearly. However, once x_{i_1} exceeds $\bar{c}_1$, i_1 and i_2 proportionally share the amount needed to compensate for i_3's contribution being below the threshold, leading to a non-linear rebate pattern. This effect arises because, beyond $\bar{c}_1$, the higher contributors collectively absorb the deficit of the lower contributor.

In Case 3, both contributors provide contributions below $\bar{c}_1$ ($x_{i_2} = 3 < \bar{c}_1$, $x_{i_3} = 2 < \bar{c}_1$), with their total contribution remaining insufficient ($x_{i_2} + x_{i_3} = 5 < c_1$). Once the provision point is reached, the rebate to i_1 under the PRT rule increases at a significantly steeper rate than under either the Shapley or PR rules. This is because, in the PRT rule, neither i_2 nor i_3 receives a rebate, so the entire surplus is allocated to i_1.

In Case 4, the contribution of i_2 and i_3 exceed the average cost ($x_{i_2} = 7 > \bar{c}_1$, $x_{i_3} = 6 > \bar{c}_1$), and their total contribution surpasses the required cost ($x_{i_2} + x_{i_3} = 13 > c_1$). Under these conditions, the PRT rule grants i_1 a rebate that increases linearly only once her contribution exceeds the threshold $\bar{c}_1$. In contrast, both the Shapley rule and the PR rule provide rebates from the very first unit of contribution. The Shapley rule redistributes the surplus according to marginal contributions, leading to moderate rebates across contributors. In contrast, the PR rule allocates the surplus proportionally among all contributors, thereby yielding smaller individual rebates.

Finally, in Case 5, where both contributions exceed $\bar{c}_1$ ($x_{i_2} = 6 > \bar{c}_1$, $x_{i_3} = 5 > \bar{c}_1$) but their combined contribution is insufficient to meet the required funding level ($x_{i_2}+x_{i_3} = 11 < c_1$), all three rules result in a full reimbursement of contributions. However, a key distinction emerges once total contributions exceed the provision point: the PRT rule ensures that contributors with contributions below $\bar{c}_1$ receive no rebate, whereas the Shapley rule and PR rule provide rebates, independent of whether individual contributions exceed $\bar{c}_1$. Under the PRT rule, the rebate stops increasing once the contributor's net payment equals the project's average cost $\bar{c}_1$. In contrast, under the Shapley and PR rules, part of i_1's contribution is rebated to i_2 and i_3.

These observations indicate that the PRT rule leads to a greater increase in the rebate received by an individual who raises their contribution, provided that the contribution exceeds $\bar{c}_1$. The PRT rule drives net payment closer to the project's average cost when the initial contribution surpasses this threshold. By contrast, the Shapley rule allocates rebates based on an individual's importance in the project's completion, thereby rebating according to systemic importance rather than overcontribution. Meanwhile, the PR rule results in a more moderate increase in the rebate received by an individual, as a significant portion of their additional contribution is redistributed to other contributors.

These observations suggest that the PRT rule effectively mitigates overpayment concerns while ensuring that high contributors receive proportionate rebates. This makes it a robust alternative for surplus rebate in public goods provision and crowdfunding mechanisms, where incentivizing large contributions while maintaining fairness is a key policy objective.

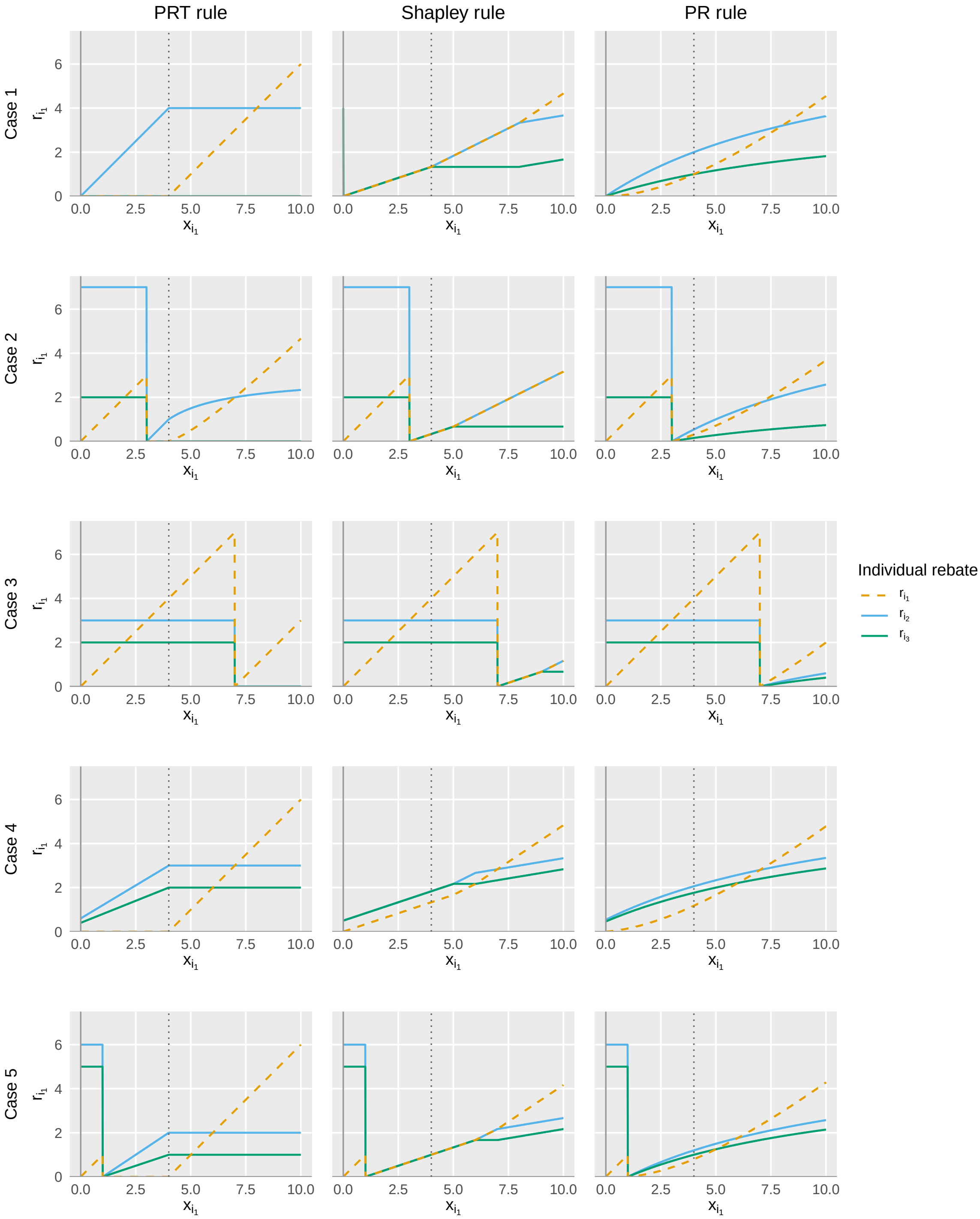

Figure 2: The figure presents rebates r_i across the five cases (rows) with individual contributions x_{i_1} on the horizontal axis. In each case, x_{i_2} and x_{i_3} are fixed, while the average cost $\bar{c}_1$ remained constant. The three columns represent the rebate rules: PRT, Shapley, and PR.

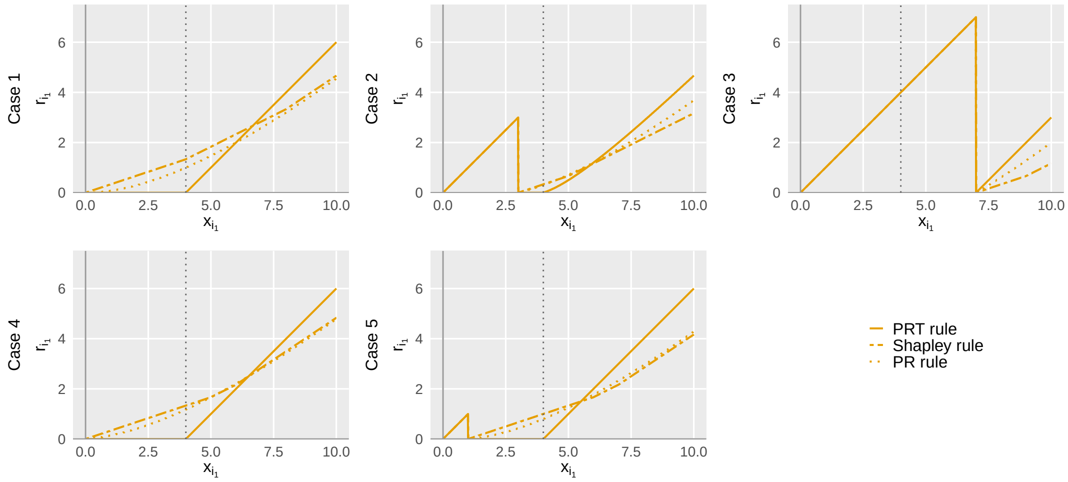

Figure 3: The figure presents rebates r_{i_1} across the five cases, with individual contributions x_{i_1} on the horizontal axis. Each plot represents the three rebate rules: PRT (solid line), Shapley (two-dash line), and PR (dotted line).

6 Concluding Remarks

This paper examines rebate problems in the provision of public goods using an axiomatic approach. We establish fundamental limitations in rebate rule design by proving that participation incentives, contribution incentives, and a fairness requirement cannot be simultaneously satisfied. To address these constraints, we introduce the PRT rule. Our analysis shows that, while satisfying weakened versions of these axioms, the PRT rule delivers higher rebates to contributors who make relatively larger contributions than those obtained under traditional benchmark rules, including the widely used Proportional Rebate rule. This suggests that the PRT rule may better support contribution incentives than existing approaches.

Provision point mechanisms aim to achieve efficient public goods provision,[16] yet often lack clearly defined rebate structures. The PRT rule addresses this gap by providing a systematic way to redistribute surplus contributions, making it applicable to a wide range of contexts, including voluntary public goods funding and crowdfunding. Beyond facilitating provision, the PRT rule may also enhance the valuation of public goods.

This paper opens avenues for both theoretical and experimental research. A natural extension of our model could incorporate reward-based or investment-based crowdfunding mechanisms into the final amount paid by contributors, thereby directly influencing their incentives. Experimentally, assessing individual contribution incentives constitutes a meaningful extension. The structure of our axioms also offers flexibility for modeling diverse market settings. In particular, our population monotonicity axiom accommodates successive rounds of contribution calls or multiple populations across different municipalities, as reflected in the formation of partitions. Another important research direction concerns the role of information: Rondeau et al. (1999) show that withholding information about provision costs can lead to contributions that more accurately reflect individuals' true valuations.

[16] A public good is efficiently provided if aggregate contributions meet or exceed the provision point (see Rondeau et al., 1999).

In this light, the PRT rule may contribute not only to surplus redistribution but also to improved demand revelation. By linking rebates to the average project cost, the rule encourages contributors to act according to their valuations while mitigating overpayment. Although the link to demand revelation is difficult to establish purely through axiomatic analysis, future laboratory or field experiments could examine whether the PRT rule effectively elicits true valuations, as envisioned in the demand revelation literature.

A Limitations Imposed by Merge-Consistency and Full Surplus Rebate

In this appendix, we consider a strengthening of Strong-Population Monotonicity, namely *Merge-Consistency*, for a class of particular problems. Merge-Consistency requires that no individual's rebate decreases when populations are merged. Unlike Population Monotonicity, introduced in Section 4, this notion focuses exclusively on rebates and does not take into account whether the funding of a more costly project alternative is permitted.

We establish a fundamental limitation that arises when Merge-Consistency is combined with Full Surplus Rebate. On a natural and economically relevant domain—namely, when no individual contributes more than the cost of the smallest non-null project alternative—these two requirements are incompatible with the funding of any positive-cost project. This result highlights that enforcing full surplus redistribution while maintaining consistency under population merging inevitably leads to the selection of the null project.

Given a problem $(I, P, \mathbf{x}) \in \Pi$, we say that *a rule φ satisfies Merge-Consistency* (M-C) in $(I, P, \mathbf{x})$ if for any partition $\mathcal{P} = \{I', I''\}$ of I such that $I' \cup I'' = I$ and $I' \cap I'' = \emptyset$:

$$\forall i \in I', \varphi_i(I, P, \mathbf{x}) \geq \varphi_i(I', P, \mathbf{x}'), \text{ and } \forall i \in I'', \varphi_i(I, P, \mathbf{x}) \geq \varphi(I'', P, \mathbf{x}''),$$

where $\mathbf{x}' \equiv (x_i)_{i \in I'}$ and $\mathbf{x}'' \equiv (x_i)_{i \in I''}$.

This axiom requires that, when two populations are merged into a single problem, no individual receives a lower rebate in the merged population than in the original one. In other words, expanding the set of contributors cannot make any participant worse off in terms of rebate.

Similarly, given a problem $(I, P, \mathbf{x}) \in \Pi$, *a rule φ satisfies* Full Surplus Rebate (FSR) in $(I, P, \mathbf{x})$ if $\sum_{i \in I} \varphi_i(I, P, \mathbf{x}) = S$.

Theorem 5. Consider a problem $(I, P, \mathbf{x})$ such that for each $i \in I, x_i < c_1$. If a rule satisfies Merge-Consistency (M-C) and Full Surplus Rebate (FSR) in $(I, P, \mathbf{x})$, then there is no funded project.

Proof. By contradiction, suppose there exists a rule φ that satisfies Merge-Consistency (M-C) and (FSR) in a problem $(I, P, \mathbf{x})$ where for each $i \in I, x_i < c_1$, and a project alternative $P_k \neq P_0$ is funded. We know that $c_k > 0$, therefore, $X - c_k < X$, meaning that there exists $i \in I$ such that $x_i > \varphi_i(I, P, \mathbf{x})$. Without loss of generality, consider a partition such that $I' = \{i\}$ and $I'' = I \backslash I'$. Since $x_i < c_1$, and φ satisfies (FSR), we know that $\varphi_i(I', P, (x_i)) = x_i$. Therefore, $\varphi_i(I', P, (x_i)) > \varphi_i(I, P, \mathbf{x})$, leading to a contradiction. ■

In practice, the condition imposed by Theorem 5 holds in many problems, where no individual contributes more than the cost of the first project alternative. If this condition is not met, the first project alternative is funded by a single agent. The implication of Theorem 5 is that the only solution to achieve full surplus rebate, ensuring independence from population size, is if no project is funded.

B Proofs

B.1 Proof of Theorem 1

Proof. We prove Theorem 1 by means of an example. Consider a problem where $I = \{i_1, i_2\}$, $P = \{P_0, P_1\}$, and c_1 with $x_{i_1} < \frac{c_1}{2}$ and $x'_{i_1} = x_{i_2} > \frac{c_1}{2}$ with $x_{i_1} + x_{i_2} > c_1$. Suppose φ satisfies (ETE), and (CM). Consider $(I, P, (x'_{i_1}, x_{i_2}))$, we have $S = 2 \times x_{i_2} - c_1$. Since φ satisfies (CM) we know that $\varphi_{i_1}(I, P, (x'_{i_1}, x_{i_2})) - \varphi_{i_1}(I, P, \mathbf{x}) \geq x_{i_2} - x_{i_1}$, meaning that $\varphi_{i_1}(I, P, (x'_{i_1}, x_{i_2})) \geq x_{i_2} - x_{i_1}$. Since φ satisfies (ETE), we know that $\varphi_{i_1}(I, P, (x'_{i_1}, x_{i_2})) = \varphi_{i_2}(I, P, (x'_{i_1}, x_{i_2}))$. Therefore, we know that $\varphi_{i_2}(I, P, (x'_{i_1}, x_{i_2})) \geq x_{i_2} - x_{i_1}$. We have $\varphi_{i_1}(I, P, (x'_{i_1}, x_{i_2})) + \varphi_{i_2}(I, P, (x'_{i_1}, x_{i_2})) \geq 2 \times x_{i_2} - 2 \times x_{i_1}$, which is a contradiction since $S = 2 \times x_{i_2} - c_1 < 2 \times x_{i_2} - 2 \times x_{i_1}$, as $x_{i_1} < \frac{c_1}{2}$. ∎

B.2 Proof of Theorem 2

Proof. **Existence:** φ_0 satisfies (ETE) and (Strong-PM). Consider $(I, P, \mathbf{x})$ with $i, j \in I$ such that $i \neq j$ and $x_i = x_j$. It is direct that φ^0 satisfies (ETE) since $\varphi_i^0(I, P, \mathbf{x}) = \varphi_j^0(I, P, \mathbf{x}) = 0$. Similarly, for any partition $\mathcal{P}$ over I such that $\mathcal{P} = \{I', I''\}$, it holds that for each $i \in I'$, $\varphi_i^0(I, P, \mathbf{x}) \geq \varphi_i^0(I', P, (x_i)) = 0$ and for each $i \in I''$, $\varphi_j^0(I, P, \mathbf{x}) \geq \varphi_j^0(I'', P, (x_j)) = 0$.
Uniqueness: If a rule satisfies (ETE) and (Strong-PM), then it has to be φ_0.
By contradiction, suppose there exists a rule φ that satisfies (ETE) and (Strong-PM) such that $\varphi \neq \varphi_0$. Since $\varphi \neq \varphi_0$, for some problem $(I, P, \mathbf{x})$, there exists $i \in I$ such that $\varphi_i(I, P, \mathbf{x}) > 0$. Theorem 3 implies that there is no rule such that $\varphi_i(I, P, \mathbf{x}) > 0$ for some $i \in I$, that satisfies (ETE) and (Strong-PM). This concludes the proof. ∎

B.3 Proof of Theorem 3

Proof. We prove Theorem 3 by means of an example. Consider a problem where $I = \{i_1, i_2\}$, $P = \{P_0, P_1\}$, $x_{i_1} > 0, x_{i_2} > 0$ and $c_1 = x_{i_1} + x_{i_2}$. Let $I' = \{i_1\}$ and $I'' = \{i_2\}$.
Suppose φ satisfies (Strong-PM) and (PSR). Consider $\varphi(I', P, (x_{i_1}))$. We know that $c_1 > x_{i_1} > 0$ and as φ satisfies (PSR), we have that $\varphi_{i_1}(I', P, (x_{i_1})) > 0$. Similarly, $\varphi(I'', P, (x_{i_2}))$, we have $c_1 > x_{i_2} > 0$ and, by (PSR), $\varphi_{i_2}(I'', P, (x_{i_2})) > 0$.
Now consider $\varphi(I'', P, \mathbf{x})$. Since $c_1 = x_{i_1} + x_{i_2}$, project alternative P_1 is realized and $S = (x_{i_1} + x_{i_2}) - c_1 = 0$. Then, $\varphi_{i_1}(I'', P, \mathbf{x}) = \varphi_{i_2}(I'', P, \mathbf{x}) = 0$. We therefore have $\varphi_{i_1}(I'', P, \mathbf{x}) \not> 0$ and $\varphi_{i_2}(I'', P, \mathbf{x}) \not> 0$ which contradict that φ satisfies (Strong-PM). ∎

B.4 Proof of Theorem 4

Proof. **Existence:** Fix an arbitrary problem $(I, P, \mathbf{x})$ and let P_{k^*} be the funded project alternative.

- ψ satisfies Equal Treatment of Equals (ETE).

Let $i, j \in I$ such that $x_i = x_j$. If $S = 0$, we know that $\psi_i(I, P, \mathbf{x}) = \psi_j(I, P, \mathbf{x}) = 0$. Similarly, if $x_i \leq \bar{c}_{k^*}$, we have $\psi_i(I, P, \mathbf{x}) = \psi_j(I, P, \mathbf{x}) = 0$. Finally, if $x_i > \bar{c}_{k^*}$, we have $\psi_i(I, P, \mathbf{x}) = (x_i - \bar{c}_{k^*}) - \frac{x_i - \bar{c}_{k^*}}{\sum_{i' \in \{i \in I : x_i > \bar{c}_{k^*}\}} x_{i'} - \bar{c}_{k^*}} \times \sum_{i' \in \{i \in I : x_i \leq \bar{c}_{k^*}\}} \bar{c}_{k^*} - x_{i'}$. It is direct that $\psi_i(I, P, \mathbf{x}) = \psi_j(I, P, \mathbf{x})$ since $x_i = x_j$.

- ψ satisfies Full Surplus Rebate (FSR).

If $S = 0$, we know that for each $i \in I$, $\psi_i(I, P, \mathbf{x}) = 0$. If $S > 0$, by construction, only individuals who have contributed an amount greater than the average cost of the project receive a refund. Let $\hat{I} \equiv \{i \in I : x_i > \overline{c}_{k^*}\}$ be the set of individuals that contribute more than $\overline{c}_{k^*}$. We have to show that $\sum_{i \in \hat{I}} \psi_i(I, P, \mathbf{x}) = S$.

$$\begin{aligned}
\sum_{i \in \hat{I}} \psi_i(I, P, \mathbf{x}) &= \sum_{i \in \hat{I}} \left((x_i - \overline{c}_{k^*}) - \frac{x_i - \overline{c}_{k^*}}{\sum_{i \in \hat{I}} x_{i'} - \overline{c}_{k^*}} \times \sum_{i' \in I \setminus \hat{I}} (\overline{c}_{k^*} - x_{i'}) \right), \\
&= \sum_{i \in \hat{I}} (x_i - \overline{c}_{k^*}) - \sum_{i' \in I \setminus \hat{I}} (\overline{c}_{k^*} - x_{i'}), \\
&= \sum_{i \in \hat{I}} (x_i - \overline{c}_{k^*}) + \sum_{i' \in I \setminus \hat{I}} (x_{i'} - \overline{c}_{k^*}), \\
&= \sum_{i \in I} x_i - c_{k^*}.
\end{aligned}$$

Since $S = \sum_{i \in I} x_i - c_{k^*}$, ψ satisfies Full Surplus Rebate (FSR).

- ψ satisfies Weak-Contribution Monotonicity (Weak-CM).

Let $i \in I$ such that $x'_i > x_i \geq \overline{c}_{k^*}$, and $P_{k'}$ be the funded project alternative in problem $(I, P, (\mathbf{x}_{-i}, x'_i))$. We have to show that

$$\psi_i(I, P, (\mathbf{x}_{-i}, x'_i)) - \psi_i(I, P, \mathbf{x}) \geq (x'_i - x_i) \times \left(1 - \frac{\check{X} \times \hat{X}_{-i}}{\left(\hat{X}_{-i} + x'_i - \overline{c}_{k^*}\right) \times \left(\hat{X}_{-i} + x_i - \overline{c}_{k^*}\right)} \right),$$

if $P_{k^*} = P_{k'}$.

By construction of ψ we know that:

$$
\begin{aligned}
&\psi_i(I,P,(\mathbf{x}_{-i},x_i'))-\psi_i(I,P,\mathbf{x})\\
&=(x_i'-\bar{c}_{k^*})-\frac{x_i'-\bar{c}_{k^*}}{\hat{X}_{-i}+x_i'-\bar{c}_{k^*}}\times\check{X}-\left((x_i-\bar{c}_{k^*})-\frac{x_i-\bar{c}_{k^*}}{\hat{X}_{-i}+x_i-\bar{c}_{k^*}}\times\check{X}\right)\\
&=x_i'-x_i-\check{X}\times\left(\frac{x_i'-\bar{c}_{k^*}}{\hat{X}_{-i}+x_i'-\bar{c}_{k^*}}-\frac{x_i-\bar{c}_{k^*}}{\hat{X}_{-i}+x_i-\bar{c}_{k^*}}\right)\\
&=x_i'-x_i-\check{X}\times\left(\frac{(x_i'-\bar{c}_{k^*})\times(\hat{X}_{-i}+x_i-\bar{c}_{k^*})-(x_i-\bar{c}_{k^*})\times(\hat{X}_{-i}+x_i'-\bar{c}_{k^*})}{(\hat{X}_{-i}+x_i'-\bar{c}_{k^*})\times(\hat{X}_{-i}+x_i-\bar{c}_{k^*})}\right)\\
&=x_i'-x_i-\check{X}\times\\
&\qquad\left(\frac{(x_i'-x_i)\times(\hat{X}_{-i}+x_i-\bar{c}_{k^*})+(x_i-\bar{c}_{k^*})\times(\hat{X}_{-i}+x_i-\bar{c}_{k^*})}{(\hat{X}_{-i}+x_i'-\bar{c}_{k^*})\times(\hat{X}_{-i}+x_i-\bar{c}_{k^*})}\right.\\
&\qquad\left.+\frac{-(x_i-\bar{c}_{k^*})\times(\hat{X}_{-i}+x_i-\bar{c}_{k^*})-(x_i'-x_i)\times(x_i-\bar{c}_{k^*})}{(\hat{X}_{-i}+x_i'-\bar{c}_{k^*})\times(\hat{X}_{-i}+x_i-\bar{c}_{k^*})}\right)\\
&=x_i'-x_i-\check{X}\times\left(\frac{(x_i'-x_i)\times(\hat{X}_{-i}+x_i-\bar{c}_{k^*}-(x_i-\bar{c}_{k^*}))}{(\hat{X}_{-i}+x_i'-\bar{c}_{k^*})\times(\hat{X}_{-i}+x_i-\bar{c}_{k^*})}\right)\\
&=(x_i'-x_i)\times\left(1-\frac{\check{X}\times\hat{X}_{-i}}{(\hat{X}_{-i}+x_i'-\bar{c}_{k^*})\times(\hat{X}_{-i}+x_i-\bar{c}_{k^*})}\right)
\end{aligned}
$$

- ψ satisfies Global Monotonicity (GM).

Let $i\in I$ such that $x_i'>x_i\geq\bar{c}_{k^*}$, and $P_{k'}$ be the funded project alternative in problem $(I,P,(\mathbf{x}_{-i},x_i'))$. We have to show that for each $j\in I$,

$$\psi_j(I,P,(\mathbf{x}_{-i},x_i'))\geq\psi_j(I,P,\mathbf{x}),$$

if $P_{k^*}=P_{k'}$.
If $x_j<\bar{c}_{k^*}$, we know that

$$\psi_j(I,P,(\mathbf{x}_{-i},x_i'))=0\geq\psi_j(I,P,\mathbf{x})=0.$$

If $x_j\geq\bar{c}_{k^*}$, we know that

$$\psi_j(I,P,\mathbf{x})=(x_j-\bar{c}_{k^*})-\frac{x_j-\bar{c}_{k^*}}{\hat{X}}\times\check{X},$$

and

$$\psi_j(I,P,(\mathbf{x}_{-i},x_i'))=(x_j-\bar{c}_{k^*})-\frac{x_j-\bar{c}_{k^*}}{\hat{X}'}\times\check{X},$$

with $\hat{X}=\sum_{i'\in\{i'\in I:x_{i'}>\bar{c}_{k^*}\}}(x_{i'}-\bar{c}_{k^*})$ and $\hat{X}'\equiv\hat{X}_{-i}+(x_i'-\bar{c}_{k^*})$. Since $x_i'>x_i\geq\bar{c}_{k^*}$ it follows that $\hat{X}'>\hat{X}$, and thus

$$\psi_j(I,P,(\mathbf{x}_{-i},x_i'))\geq\psi_j(I,P,\mathbf{x}),$$

which concludes the proof.
Uniqueness: We fix a problem $(I,P,\mathbf{x})$ with funded project alternative P_k. Suppose that φ satisfies ETE, FSR, Weak-CM and GM.

Let $i \in I$ be such that $x_i \geq \overline{c}_k$, and let $x_i' > x_i$ and $P_{k'}$ is the funded project alternative in $(I, P, (\mathbf{x}_i, x_i'))$. Since φ satisfies Weak-CM, we have

$$\varphi_i(I, P, (\mathbf{x}_{-i}, x_i')) - \varphi_i(I, P, \mathbf{x}) \geq (x_i' - x_i) \times \left(1 - \frac{\check{X} \times \hat{X}_{-i}}{\left(\hat{X}_{-i} + x_i' - \overline{c}_{k^*}\right) \times \left(\hat{X}_{-i} + x_i - \overline{c}_{k^*}\right)}\right).$$

We have shown that for the PRT rule ψ, we have

$$\psi_i(I, P, (\mathbf{x}_{-i}, x_i')) - \psi_i(I, P, \mathbf{x}) = (x_i' - x_i) \times \left(1 - \frac{\check{X} \times \hat{X}_{-i}}{(\hat{X}_{-i} + x_i' - \overline{c}_{k^*}) \times (\hat{X}_{-i} + x_i - \overline{c}_{k^*})}\right).$$

Comparing the two expressions yields

$$\varphi_i(I, P, (\mathbf{x}_{-i}, x_i')) - \varphi_i(I, P, \mathbf{x}) \geq \psi_i(I, P, (\mathbf{x}_{-i}, x_i')) - \psi_i(I, P, \mathbf{x}),$$

for every individual $i \in I$ with $x_i \geq \overline{c}_k$ and every $x_i' > x_i$ that leaves the funded project at P_k. By relabelling agents, the same inequality holds for every agent in I whose contribution is at least $\overline{c}_k$.

Consider now the contribution vector $\overline{\mathbf{x}}$ defined by $\overline{x}_i = \overline{c}_k$ for each $i \in I$. In this profile, the funded project is P_k, and the surplus is zero:

$$\sum_{i \in I} \overline{x}_i = c_k \text{ and } S = 0.$$

Since φ satisfies FSR, we have

$$\sum_{i \in I} \varphi_i(I, P, \overline{\mathbf{x}}) = 0,$$

rebates are non-negative, and ETE implies that all individuals receive the same rebate at $\overline{\mathbf{x}}$, hence for each $i \in I$, $\varphi_i(I, P, \overline{\mathbf{x}}) = 0$. Similarly, under the PRT rule ψ, it also follows that for each $i \in I$, $\psi_i(I, P, \overline{\mathbf{x}}) = 0$.

Consider now an individual $i \in I$ such that $x_i \geq \overline{c}_k$ in the orignal problem $(I, P, \mathbf{x})$. Consider a hypothetical path of contribution levels for individual i, keeping all other contributions at $\overline{c}_k$, from $\overline{x}_i$ to x_i. It follows from Weak-CM that,

$$\varphi_i(I, P, (\overline{\mathbf{x}}_{-i}, x_i)) - \varphi_i(I, P, \overline{\mathbf{x}}) \geq \psi_i(I, P, (\overline{\mathbf{x}}_{-i}, x_i)) - \psi_i(I, P, \overline{\mathbf{x}}),$$

that is

$$\varphi_i(I, P, (\overline{\mathbf{x}}_{-i}, x_i)) \geq \psi_i(I, P, (\overline{\mathbf{x}}_{-i}, x_i)).$$

Moreover, since φ satisfies GM, increasing the contribution of i from $\overline{x}_i = \overline{c}_k$ to x_i cannot reduce any other individual's rebate. Therefore, for all $j \neq i$,

$$\varphi_j(I, P, (\overline{\mathbf{x}}_{-i}, x_i)) \geq \varphi_j(I, P, \overline{\mathbf{x}}) = 0 = \psi_j(I, P, \overline{\mathbf{x}}).$$

We then repeat this argument for each i' such that $x_{i'} \geq \overline{c}_k$. It follows that, when we move from $(I, P, (\overline{\mathbf{x}}_{-i}, x_i))$ to $(I, P, (\overline{\mathbf{x}}_{-\{i,i'\}}, x_i, x_{i'}))$, we have

$$\varphi_i(I, P, (\overline{\mathbf{x}}_{-\{i,i'\}}, x_i, x_{i'})) \geq \varphi_i(I, P, (\overline{\mathbf{x}}_{-i}, x_i)),$$

and

$$\varphi_{i'}(I, P, (\overline{\mathbf{x}}_{-\{i,i'\}}, x_i, x_{i'})) - \varphi_{i'}(I, P, (\overline{\mathbf{x}}_{-i}, x_i)) \geq \psi_{i'}(I, P, (\overline{\mathbf{x}}_{-\{i,i'\}}, x_i, x_{i'})) - \psi_{i'}(I, P, (\overline{\mathbf{x}}_{-i}, x_i)).$$

We arrive at a vector $(\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})$ where $\hat{I} \equiv \{i \in I : x_i \geq \overline{c}_k\}$ and $\check{I} \equiv \{j \in I : x_j < \overline{c}_k\}$, at which

$$\forall i \in \hat{I}, \varphi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) \geq \psi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})),$$

and

$$\forall j \in \check{I}, \varphi_j(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) \geq \psi_j(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) = 0.$$

Now sum these inequalities over all $i \in \hat{I}$. Under the PRT rule, individuals in $\check{I}$ receive no rebate, so

$$\sum_{i \in \hat{I}} \psi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) = S = \sum_{i \in I} \psi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})).$$

Since φ satisfies FSR, we have

$$\sum_{i \in I} \varphi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) = S.$$

Rebates are weakly positive so,

$$\sum_{i \in I} \varphi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) \geq \sum_{\hat{i} \in \hat{I}} \varphi_{\hat{i}}(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})).$$

Combining these inequalities, we obtain

$$S \geq \sum_{\hat{i} \in \hat{I}} \varphi_{\hat{i}}(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) \geq \sum_{i \in I} \psi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) = S.$$

Thus, for each $i \in \hat{I}$, $\varphi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) = \psi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}}))$.
Moreover, all individuals $j \in \check{I}$ must receive zero rebate under φ, because their rebates are weakly positive and the entire surplus S is already exhausted by individuals in $\hat{I}$. Since PRT also assigns zero rebate to these under-contributors, we conclude that

$$\forall i \in I, \varphi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) = \psi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})). \tag{1}$$

We now relate $(\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})$ to $\mathbf{x}$. Consider an individual $j \in \check{I}$, so $x_j \leq \overline{c}_k$ in the original vector $\mathbf{x}$. Under the PRT rule, such an individual has no excess contribution, so

$$\psi_j(I, P, \mathbf{x}) = 0.$$

Suppose, for a contradiction, that $\varphi_j(I, P, \mathbf{x}) > 0$. From $\mathbf{x}$ to $(\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})$, we increase only the contributions of the individuals in $\check{I}$, from x_j up to $\overline{c}_k$ for each $j \in \check{I}$ while the contributions of individuals in $\hat{I}$ are unchanged. By GM, increasing contributions cannot decrease any individual's rebate, so in particular

$$\varphi_j(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) \geq \varphi_j(I, P, \mathbf{x}) > 0.$$

However, from (1) and the definition of ψ, we know that for each $j \in \check{I}$,

$$\varphi_j(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) = 0 = \psi_j(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})),$$

a contradiction. Thus, for each $j \in \check{I}$,

$$\varphi_j(I, P, \mathbf{x}) = 0 = \psi_j(I, P, \mathbf{x}). \tag{2}$$

Therefore, we know that

$$\sum_{i \in \hat{I}} \varphi_i(I, P, \mathbf{x}) = \sum_{i \in I} \varphi_i(I, P, \mathbf{x}) = S, \tag{3}$$

since φ satisfies FSR. Similarly, under the PRT rule,

$$\sum_{i \in \hat{I}} \psi_i(I, P, \mathbf{x}) = \sum_{i \in I} \psi_i(I, P, \mathbf{x}) = S. \tag{4}$$

Suppose, for contradiction that for $i \in \hat{I}$, we have $\varphi_i(I, P, \mathbf{x}) \neq \psi_i(I, P, \mathbf{x})$. Since the sums over $\hat{I}$ coincide by (3)-(4), there must exist $i, i' \in \hat{I}$ with

$$\varphi_i(I, P, \mathbf{x}) > \psi_i(I, P, \mathbf{x}) \text{ and } \varphi_{i'}(I, P, \mathbf{x}) < \psi_{i'}(I, P, \mathbf{x}). \tag{5}$$

Consider again the transformation from $\mathbf{x}$ to $(\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})$, in which we only increase the contributions of individuals in $\check{I}$ from their original levels $x_j \leq \overline{c}_k$ up to $\overline{c}_k$. Under the PRT rule, this transformation leaves the contributions of all individuals in $\hat{I}$ unchanged, and therefore preserves their total excess. The only effect is an increase in the total surplus.
Because the excess of the agents in $\hat{I}$ does not change when the contributions of under-contributors are raised from below $\overline{c}_k$ to exactly $\overline{c}_k$, the PRT rebate of each agent in $\hat{I}$ at the vector $(\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})$ is obtained from her rebate at $\mathbf{x}$ by a positive affine rescaling. In particular, for every $i \in \hat{I}$,

$$\psi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) > \psi_i(I, P, \mathbf{x}). \tag{6}$$

By GM, the transformation from $\mathbf{x}$ to $(\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})$ weakly increases all contributions, and therefore cannot reduce any individual's rebate under φ. Hence, for all $i \in I$,

$$\varphi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) \geq \varphi_i(I, P, \mathbf{x}). \tag{7}$$

Combining (5), (6), and (7) with (1), we obtain for individual i:

$$\psi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) = \varphi_i(I, P, (\overline{\mathbf{x}}_{-\hat{I}}, \mathbf{x}_{-\check{I}})) \geq \varphi_i(I, P, \mathbf{x}) > \psi_i(I, P, \mathbf{x}),$$

and similarly for i'

$$\psi_{i'}(I,P,(\overline{\mathbf{x}}_{-\hat{I}},\mathbf{x}_{-\check{I}})) = \varphi_{i'}(I,P,(\overline{\mathbf{x}}_{-\hat{I}},\mathbf{x}_{-\check{I}})) \geq \varphi_{i'}(I,P,\mathbf{x}) < \psi_{i'}(I,P,\mathbf{x}).$$

Inequalities (7) and the strict inequalities in (5)-(6) imply that at $(\overline{\mathbf{x}}_{-\hat{I}},\mathbf{x}_{-\check{I}})$ we would still have at least one individual $\hat{i} \in \hat{I}$ with $\varphi_{\hat{i}}(I,P,(\overline{\mathbf{x}}_{-\hat{I}},\mathbf{x}_{-\check{I}})) \neq \psi_{\hat{i}}(I,P,(\overline{\mathbf{x}}_{-\hat{I}},\mathbf{x}_{-\check{I}}))$, contradicting (1). Therefore, for each $i \in \hat{I}$,

$$\varphi_i(I,P,\mathbf{x}) = \psi_i(I,P,\mathbf{x}).$$

Together with (2), this yields for each $i \in I$,

$$\varphi_i(I,P,\mathbf{x}) = \psi_i(I,P,\mathbf{x}).$$

Since $(I,P,\mathbf{x})$ was an arbitrary problem, we conclude that $\varphi = \psi$ on Π. ■

B.5 Proof of Proposition 1

Proof. Fix an arbitrary problem $(I,P,\mathbf{x})$ and let P_{k^*} be the funded project alternative.

- ψ satisfies Population Monotonicity (PM).

By the definition of (PM), we only need to consider the partitions in which P_k is funded and $k = k^*$. Suppose by contradiction that there exists $i \in I$ such that, without loss of generality $i \in I'$, and $\psi_i(I',P,\mathbf{x}') > 0$ and $\psi_i(I,P,\mathbf{x}) = 0$. We know that the project alternative P_{k^*} is funded in $(I',P,\mathbf{x}')$. Let $\overline{c}'_{k^*} \equiv \frac{c_{k^*}}{|I'|}$. It follows that $\overline{c}'_{k^*} \geq \overline{c}_{k^*}$. If $x_i \leq \overline{c}'_{k^*}$, we know that $\psi_i(I',P,\mathbf{x}') = 0$ which contradict $\psi_i(I',P,\mathbf{x}') > 0$. Suppose $x_i > \overline{c}'_{k^*}$. Since P_{k^*} is funded, and $\psi_i(I',P,\mathbf{x}') > 0$, we know that $S' > 0$.

Claim 1. If $S > 0$ and $x_i > \overline{c}_{k^*}$, then $\psi_i(I,P,\mathbf{x}) > 0$.

Proof. By definition,

$$\begin{aligned}\psi_i(I,P,\mathbf{x}) &= (x_i - \overline{c}_{k^*}) - \frac{x_i - \overline{c}_{k^*}}{\hat{X}} \times \check{X}, \\ &= (x_i - \overline{c}_{k^*}) \times \left(1 - \frac{\check{X}}{\hat{X}}\right).\end{aligned}$$

Since $S > 0$ we know that $\hat{X} > \check{X}$ implying that $\frac{\check{X}}{\hat{X}} < 1$. Therefore, $\left(1 - \frac{\check{X}}{\hat{X}}\right) > 0$. As $x_i > c_{k^*}$, it follows that $\psi_i(I,P,\mathbf{x}) > 0$. ■

We know that $S \geq S'$, therefore, $S > 0$. Since $x_i > \overline{c}'_{k^*} \geq \overline{c}_{k^*}$, by Claim 1 we have that $\psi_i(I,P,\mathbf{x}) > 0$.

- ψ satisfies the Monotonicity of Net Payment (MNP).

Suppose $i,j \in I$, without loss of generality, assume that $x_i \geq x_j$. We have to show that $x_i - \psi_i(I,P,\mathbf{x}) \geq x_j - \psi_j(I,P,\mathbf{x})$.

- **Case 1:** If $S = 0$. Then for each $i' \in I$, $\psi_{i'}(I, P, \mathbf{x}) = 0$. Since $x_i \geq x_j$ we have that $x_i - \psi_i(I, P, \mathbf{x}) \geq x_j - \psi_j(I, P, \mathbf{x})$.
- **Case 2:** If $S > 0$, $\bar{c}_{k^*} \geq x_i \geq x_j$. Then $\psi_i(I, P, \mathbf{x}) = \psi_j(I, P, \mathbf{x}) = 0$. Since $x_i \geq x_j$ we have that $x_i - \psi_i(I, P, \mathbf{x}) \geq x_j - \psi_j(I, P, \mathbf{x})$.
- **Case 3:** If $S > 0$, $x_i \geq \bar{c}_{k^*} \geq x_j$. Then $\psi_j(I, P, \mathbf{x}) = 0$. We have to show that $x_i - \psi_i(I, P, \mathbf{x}) \geq \bar{c}_{k^*}$. It follows that

$$(x_i - \bar{c}_{k^*}) \geq (x_i - \bar{c}_{k^*}) - \frac{x_i - \bar{c}_{k^*}}{\sum_{i' \in \{i \in I : x_i > \bar{c}_{k^*}\}} x_{i'} - \bar{c}_{k^*}} \times \sum_{i' \in \{i \in I : x_i \leq \bar{c}_{k^*}\}} (\bar{c}_{k^*} - x_{i'}). \qquad (8)$$

  Since $x_i - \psi_i(I, P, \mathbf{x}) \geq x_i - (x_i - \bar{c}_{k^*})$ by equation (8), it is direct that $x_i - (x_i - \bar{c}_{k^*}) \geq \bar{c}_{k^*}$, and therefore $x_i - \psi_i(I, P, \mathbf{x}) \geq x_j - \psi_j(I, P, \mathbf{x})$.
- **Case 4:** If $S > 0$, $x_i \geq x_j \geq \bar{c}_{k^*}$. We have to show that:

$$x_i \geq x_j - \psi_j(I, P, \mathbf{x}) + \psi_i(I, P, \mathbf{x}).$$

$$\begin{aligned}
& x_j + \psi_i(I, P, \mathbf{x}) - \psi_j(I, P, \mathbf{x}) \\
&\quad = x_j + (x_i - \bar{c}_{k^*}) - \frac{x_i - \bar{c}_{k^*}}{\sum\limits_{i' \in \{i \in I : x_i > \bar{c}_{k^*}\}} x_{i'} - \bar{c}_{k^*}} \times \sum_{i' \in \{i \in I : x_i \leq \bar{c}_{k^*}\}} (\bar{c}_{k^*} - x_{i'}) \\
&\quad\quad - \left((x_i - \bar{c}_{k^*}) - \frac{x_i - \bar{c}_{k^*}}{\sum\limits_{i' \in \{i \in I : x_i > \bar{c}_{k^*}\}} x_{i'} - \bar{c}_{k^*}} \times \sum_{i' \in \{i \in I : x_i \leq \bar{c}_{k^*}\}} (\bar{c}_{k^*} - x_{i'}) \right) \\
&\quad = x_j + x_i - \bar{c}_{k^*} - x_j + \bar{c}_{k^*} \\
&\quad\quad - \left(\frac{x_i - \bar{c}_{k^*} - x_j + \bar{c}_{k^*}}{\sum\limits_{i' \in \{i \in I : x_i > \bar{c}_{k^*}\}} x_{i'} - \bar{c}_{k^*}} \right) \times \sum_{i' \in \{i \in I : x_i \leq \bar{c}_{k^*}\}} (\bar{c}_{k^*} - x_{i'}) \\
&\quad\quad x_i - \left(\frac{x_i - x_j}{\sum\limits_{i' \in \{i \in I : x_i > \bar{c}_{k^*}\}} x_{i'} - \bar{c}_{k^*}} \right) \times \sum_{i' \in \{i \in I : x_i \leq \bar{c}_{k^*}\}} (\bar{c}_{k^*} - x_{i'}) \\
&\quad = x_i - \left(\frac{x_i - x_j}{\sum\limits_{i' \in \{i \in I : x_i > \bar{c}_{k^*}\}} x_{i'} - \bar{c}_{k^*}} \right) \times \sum_{i' \in \{i \in I : x_i \leq \bar{c}_{k^*}\}} (\bar{c}_{k^*} - x_{i'})
\end{aligned}$$

  Since $x_i \geq x_j$ we know that $\frac{x_i - x_j}{\sum_{i' \in \{i \in I : x_i > \bar{c}_{k^*}\}} x_{i'} - \bar{c}_{k^*}} \geq 0$. Similarly, $\sum_{i' \in \{i \in I : x_i \leq \bar{c}_{k^*}\}} (\bar{c}_{k^*} - x_{i'}) \geq 0$. We therefore have

$$x_i \geq x_i - \left(\frac{x_i - x_j}{\sum_{i' \in \{i \in I : x_i > \bar{c}_{k^*}\}} x_{i'} - \bar{c}_{k^*}} \right) \times \sum_{i' \in \{i \in I : x_i \leq \bar{c}_{k^*}\}} (\bar{c}_{k^*} - x_{i'}).$$

  This implies that $x_i \geq x_j - \psi_j(I, P, \mathbf{x}) + \psi_i(I, P, \mathbf{x})$.

■

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